§ 3.6 – Variation of parameters

Recall: To find a particular solution to
$$y'' + p(x)y' + q(x)y = g(x),$$
one method is undetermined coefficients. This method works well if $g(x) = \exp(\alpha x)$, $\cos(\beta x)$ or $\sin(\beta x)$, and/or polynomials. It also works for products / sums of this type; it doesn't work for other types.

Ex: $y'' + 4y = 3\csc t$. ← we don't expect homogeneous coords to work well here!

Solution: Use variation of parameters!

Idea: Try to find functions $u_1(x)$ & $u_2(x)$ such that
$$u_1(x)y_1(x) + u_2(x)y_2(x)$$
is a solution to the non-homogeneous eq. (where y_1 & y_2 are solutions to the homogeneous eq.).

↳ Doing this is (a) tedious, (b) time-consuming, and (c) pretty not-intuitive. However, it DOES work!

Ex: For the ODE above $[y'' + 4y = 3\csc t]$, v.o.p. yields that

$y'' + 4y = 0 \Rightarrow r^2 + 4 = 0$
$\Rightarrow r = \pm 2i$

$y_1 = \cos(2x) \quad y_2 = \sin(2x)$

$u_1 y_1 + u_2 y_2$ is a particular sol'n when

$u_1 = -3\sin t + c_1$ &

$u_2 = \frac{3}{2}\ln|\csc t - \cot t| + 3\cos t + c_2$.

Instead of going through the details, we skip to the punchline!

Thm: If p, q, g are continuous on an open interval I & if y_1 & y_2 are a fund. sys. of sol'ns of the homogeneous ODE $y'' + p(x)y' + q(x)y = 0$, then a particular solution of

$$y'' + p(x)y' + q(x)y = g(x)$$

is $Y = -y_1(t) \cdot \int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(y_1,y_2)(s)}\,ds,$

where t_0 is any pt in I.

Note: Either I give interval or you use $t_0 = 0$!
& you pick a convenient t_0 in it

Ex: $y'' + y = \tan x \rightarrow 0 < t < \frac{\pi}{2}$

↳ Hom: $r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow y_1 = \cos(x)$ & $y_2 = \sin(x)$.

- y_1 & y_2 F.S.S? : ∘ $y_1'' + y_1 = -\cos x + \cos x = 0$ & $y_2'' + y_2 = -\sin(x) + \sin(x) = 0$

  so y_1 & y_2 both sol'ns to homogeneous. $y'' + y = 0$.

  ∘ $W(y_1, y_2) = \det\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$.

  → y_1 & y_2 are F.S.S.!

- Part. sol'n: $Y = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(y_1,y_2)(s)}\,ds$

  where • $y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$, $g(x) = \tan x$, $I = (0, \pi/2)$, $t_0 =$ any pt in I. $W(y_1,y_2)(s) = 1$

↳ $Y = -\cos(t)\int_{t_0}^{t} \frac{\sin(s)\tan(s)}{1}\,ds + \sin(t)\int_{t_0}^{t} \frac{\cos(s)\tan(s)}{1}\,ds$

$= -\cos(t)\int_{t_0}^{t} \frac{\sin^2(s)}{\cos(s)}\,ds + \sin(t)\int_{t_0}^{t} \sin(s)\,ds \longrightarrow$

$$= -\cos(t)\int_{t_0}^{t} \frac{\sin^2(s)}{\cos(s)}\,ds + \sin(t)\int_{t_0}^{t} \sin(s)\,ds \qquad \text{(just recopying)}$$

$$= -\cos(t)\int_{t_0}^{t} \frac{(1-\cos^2(s))}{\cos s}\,ds + \sin(t)\left(-\cos(s)\Big]_{s=t_0}^{s=t}\right)$$

$$= -\cos(t)\int_{t_0}^{t} \sec(s) - \cos(s)\,ds + \sin(t)\left(-\cos(t) + \cos(t_0)\right)$$

$$= -\cos(t)\left(\ln|\sec(s)+\tan(s)| - \sin(s)\Big]_{s=t_0}^{s=t}\right) + \sin(t)\left(-\cos(t)+\cos(t_0)\right)$$

$$= -\cos(t)\Big(\ln(\sec(t)+\tan(t)) - \sin(t) - \ln(\sec(t_0)+\tan(t_0)) + \sin(t_0)\Big) + \sin(t)\left(-\cos(t)+\cos(t_0)\right) \qquad t\in\left(0,\tfrac{\pi}{2}\right)$$

• Pick $t_0 \in \left(0, \frac{\pi}{2}\right)$: I pick $\frac{\pi}{4}$.

$$\hookrightarrow Y(t) = -\cos(t)\Big(\ln(\sec(t)+\tan(t)) - \sin(t) - \ln\big(\underbrace{\sec(\tfrac{\pi}{4})}_{\frac{1}{1/\sqrt{2}} = \sqrt{2}} + \underbrace{\tan(\tfrac{\pi}{4})}_{1}\big) + \underbrace{\sin(\tfrac{\pi}{4})}_{\frac{1}{\sqrt{2}}}\Big) + \sin(t)\big(-\cos(t) + \underbrace{\cos(\pi/4)}_{\frac{1}{\sqrt{2}}}\big)$$

Gen Soln:

$$y = c_1 y_1 + c_2 y_2 + Y(t) \quad \text{where } y_1 = \cos(t) \ \& \ y_2 = \sin(t).$$

Ex: Show that $y_1 = t^2$ & $y_2 = t^{-1}$ satisfy the corresponding hom. eq. to $t^2y'' - 2y = 3t^2 - 1$ and find a particular sol'n to nonhom. eq. Assume $t > 0$.

$\underbrace{t>0}_{I=(0,\infty)}$

$y_2' = -t^{-2}$

$y_2'' = 2t^{-3}$

- $t^2 y_1'' - 2y_1 = t^2(2) - 2t^2 = 0$

$$t^2(y_2'') - 2y_2 = t^2\left(2t^{-3}\right) - 2t^{-1}$$

$$= 2t^{-1} - 2t^{-1} = 0.$$

- $W(y_1, y_2) = \det\begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix} = -1 - 2 = -3.$

- <u>Part. sol'n</u>:

$$Y = -t^2 \int_{t_0}^{t} \frac{s^{-1}\cdot(3s^2-1)}{-3}\,ds + t^{-1}\int_{t_0}^{t} \frac{s^2(3s^2-1)}{-3}\,ds$$

$$= \frac{-t^2}{-3}\int_{t_0}^{t} 3s - s^{-1}\,ds + \frac{t^{-1}}{-3}\int_{t_0}^{t} 3s^4 - s^2\,ds$$

$$= \frac{t^2}{3}\left(\frac{3}{2}s^2 - \ln(s)\Big]_{t_0}^{t}\right) - \frac{t^{-1}}{3}\left(\frac{3}{5}s^5 - \frac{1}{3}s^3\Big]_{t_0}^{t}\right)$$

$\leftarrow$ t_0 any # in $(0,\infty)$; I pick 1.

$$= \frac{t^2}{3}\left(\frac{3}{2}t^2 - \ln(t) - \frac{3}{2} + 0\right) - \frac{t^{-1}}{3}\left(\frac{3}{5}t^5 - \frac{1}{3}t^3 - \frac{3}{5} + \frac{1}{3}\right)$$

$$= \frac{t^4}{2} - \frac{t^2\ln(t)}{3} - \frac{t^2}{2} - \frac{t^4}{5} + \frac{t^2}{9} + \frac{t^{-1}}{5} + \frac{t^{-1}}{9}.$$