§3.5 - Nonhomogeneous Equations & Undetermined Coefficients

The goal will be to solve ODEs of the form

$$y'' + p(x)\,y' + q(x)\,y = g(x) \qquad (*)$$

where $g(x) \neq 0$.

↳ Note: Corresponding to $(*)$ is the homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0. \qquad (**).$$

- As it happens, step 1 is to solve the homogeneous eq $(**)$, as indicated by the following theorem.

Thm: The general solution of the ODE $(*)$ has the form

$$y = C_1 y_1 + C_2 y_2 + Y(t),$$

where y_1 & y_2 are a fundamental system for the homogeneous ODE $(**)$ & where Y is some specific solution to the nonhomogeneous ODE $(*)$. aka: a "particular solution"

↳ To find gen. sol of $(*)$, do 3 things:

1. Find $c_1 y_1 + c_2 y_2$ from homogeneous $(**)$
2. Find a solution to $(*)$ (aka, find a particular solution)
3. Add ① & ② together.

Because ① was covered in §3.1, §3.3, & §3.4, we focus on ② using the method of undetermined coefficients.

Idea: Guess (vaguely) what $Y(x)$ may look like, given $g(x)$, find Y', Y'', and plug into ODE to solve for missing coefficients.

Ex: $y''-3y'-4y=3e^{2x}$. Homogeneous: $y''-3y'-4y=0 \iff (r-4)(r+1)=0 \iff r=4$ or $r=-1 \Rightarrow$ Fund Sys $=\{e^{4x}, e^{-x}\}$.

- Goal: Find Y s.t. $Y''-3Y'-4Y=3e^{2x}$.
- Guess: B/c $e^{\square}$ reproduces itself w/ derivatives, we guess that $Y=$ some multiple of e^{2x}: $Y=Ae^{2x}$.
- Take derivatives & find A:

$$Y'=2Ae^{2x} \Rightarrow Y''=4Ae^{2x}$$

$$\hookrightarrow 4Ae^{2x}-3(2Ae^{2x})-4(Ae^{2x})=3e^{2x}$$

$$\Rightarrow -6Ae^{2x}=3e^{2x} \Rightarrow -6A=3 \Rightarrow A=-1/2$$

- Plug into Y: $Y(x)=-\frac{1}{2}e^{2x}$.
- Gen Soln: $y=c_1e^{4x}+c_2e^{-x}-\frac{1}{2}e^{2x}$.

Ex: $y'' - 3y' - 4y = 2\sin x.$

Guess 1: $Y = A\sin x.$ $\left(\Rightarrow Y' = A\cos x \Rightarrow Y'' = -A\sin x\right)$

$\hookrightarrow -A\sin x - \underline{3A\cos x} - 4A\sin x = 2\sin x$

$\Rightarrow -5A\sin x - 3A\cos x = 2\sin x$

$\Rightarrow 2\sin x + 5A\sin x + 3A\cos x = 0.$

$\Rightarrow (2+5A)\sin x + 3A\cos x = 0.$

Has to be true for $\underline{\underline{\text{ALL}}}$ x!

$\hookrightarrow$ This is hard to solve, but we can plug in points:

- @ $x = \frac{\pi}{2}$: $2 + 5A = 0 \Rightarrow A = \frac{-5}{2}.$
- @ $x = 0$: $3A = 0 \Rightarrow A = 0.$
- (any others you want to check, e.g. @ $\frac{\pi}{4} \Rightarrow A = -1, \ldots$)

This term seems plausible since $(\sin x)' = \cos x$; see here

Because no one A value works for $x = \frac{\pi}{2}, x = 0, x = \frac{\pi}{4}, \ldots,$ there's no A that works for $\underline{\underline{\text{ALL}}}$ x.

Guess 2: $Y = A\sin x + B\cos x$

$Y' = A\cos x - B\sin x \Rightarrow Y'' = -A\sin x - B\cos x$

$\hookrightarrow (-A\sin x - B\cos x) - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$

$$\Longleftrightarrow \sin x\underbrace{(-A + 3B - 4A)}_{-5A+3B} + \cos x\underbrace{(-B - 3A - 4B)}_{-5B-3A} = 2\sin x$$

$\Longleftrightarrow -5A + 3B = 2$ & $\underline{-5B - 3A} = 0$

RHS has 2 sinx's; sinx's on LHS; RHS has 0 cosx's; cosx's on LHS

- Solve for A & B: $A = \frac{-5}{17}$ & $B = \frac{3}{17}$

$\Rightarrow Y(x) = \frac{-5}{17}\sin x + \frac{3}{17}\cos x$ solves this ODE.

Ex: $y'' - 3y' - 4y = 4x^2 - 1.$

Guess: $Y = Ax^2 + Bx + C$

$\hookrightarrow y' = 2Ax + B, \quad Y'' = 2A$

$\Rightarrow 2A - 3(2Ax + B) - 4(Ax^2 + Bx + C) = 4x^2 - 1.$

$\Rightarrow x^2(-4A) + x(-6A - 4B) + (2A - 3B - 4C) = 4x^2 - 1$

$\hookrightarrow$ • $-4A = 4 \Rightarrow \boxed{A = -1.}$

• $-6A - 4B = 0 \Rightarrow -6(-1) - 4B = 0$

$\Rightarrow 6 - 4B = 0$

$\Rightarrow \boxed{B = \frac{3}{2}.}$

• $2A - 3B - 4C = \cancel{4} - 1 \Rightarrow 2(-1) - 3(3/2) - 4C = -1$

$\Rightarrow -2 - \frac{9}{2} - 4C = -1$

$\Rightarrow 2 + \frac{9}{2} - 1 = -4C$

$\Rightarrow \frac{11}{2} = -4C \Rightarrow \boxed{C = \frac{-11}{8}}$

So, $Y = -x^2 + \frac{3}{2}x - \frac{11}{8}.$

To summarize:

If $g(x)$ is...	... then you should guess....
① e^{ax}	$Y = Ae^{ax}$
② $\sin\beta x$ or $\cos\beta x$	$Y = A\sin\beta x + B\cos\beta x$
③ Polynomial of degree n (e.g. x^3-2x+4)	Polynomial of same degree (e.g. Ax^3+Bx^2+Cx+D).

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... but even then, there are exceptions.

Ex: $y''-3y'-4y=2e^{-x}$.

Guess: $Y=e^{-x}$ $(\Rightarrow Y'=-e^{-x}$ & $Y''=e^{-x})$

↳ $e^{-x}+3e^{-x}-4e^{-x}=2e^{-x} \Rightarrow 0=2e^{-x}$. (impossible!)

Note: e^{-x} is a solution to the homogeneous eq.

$y''-3y'-4y=0$

$\left(r^2-3r-4=0 \Rightarrow (r-4)(r+1)=0 \Rightarrow r=4 \quad r=-1 \Rightarrow e^{4x} \quad e^{-x}\right)$

What do we do?

↳ If the Y you guess is a solution to the homogeneous eq., let guess #2 be $x \cdot Y$. (Cont'd on pg 9)

Ex: Pick a $Y(x)$ "guess" corresponding to each $g(x)$ below.

① $g(x) = e^{2x}$ ② $g(x) = \sin(2x)$ ③ $g(x) = x^5 - 1$

④ $g(x) = 5\cos x + \sin x$

⑤ $g(x) = x^2 e^{-3x}$ ⑥ $g(x) = 4$

⑦ $g(x) = e^{4x} + x^2 e^x + \cos(3x) - x\sin(x)$.

① $Y = Ae^{2x}$ ② $Y = A\sin(2x) + B\cos(2x)$ ③ $Y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$

④ $Y = A\cos(x) + B\sin(x)$ ⑤ Here, we have $\begin{pmatrix}\text{polynomial} \\ \text{deg } 2\end{pmatrix} * e^{-3x}$, so we guess $Y = (Ax^2 + Bx + C)e^{-3x}$

This works when we have a "product of two guesses": Guess a thing that's a product of what you would have guessed!

⑥ $Y = A$.

⑦ When we have a "<u>sum</u> of guesses", we have to do something different!

↳ For LHS $= \overset{①}{e^{4x}} + \overset{②}{x^2 e^x} + \overset{③}{\cos(3x)} - \overset{④}{x\sin x}$, you have to do <u>one guess for every summand</u>!

↳ ① $e^{4x} \longrightarrow$ Guess $Y = Ae^{4x}$ & Consider LHS $= Ae^{4x}$

② $x^2 e^x \longrightarrow$ Guess: $Y = (Ax^2 + Bx + C)e^x$ & consider LHS = that

③ $\cos(3x) \longrightarrow$ Guess: $Y = A\cos(3x) + B\sin(3x)$ . . .

④ $-x\sin x \longrightarrow$ Guess: $(Ax+B)\sin(x) + (Cx+D)\cos(x)$.

Ex: Find the general solution of

§3.5 #10

$$y'' + y = 3\sin(2t) + t\cos(2t).$$

① Homogeneous: $y''+y=0 \xrightarrow[\text{eq}]{\text{char}} r^2+1=0 \longrightarrow r = \pm i = 0 \pm i$

$$\Rightarrow e^{0t}(c_1\cos(t) + c_2\sin(t))$$
$$= c_1\cos(t) + c_2\sin(t).$$

$$\boxed{\begin{array}{l}\text{Ans} \\ y = c_1\cos(t) + c_2\sin(t) \\ + \left(\tfrac{4}{3}t - \tfrac{52}{9}\right)\sin(2t) \\ + \left(t - \tfrac{16}{3}\right)\cos(2t).\end{array}}$$

② Nonhomogeneous:

$$y'' + y = 3\sin(2t) + t\cos(2t).$$

Guess: $Y = (At+B)\sin(2t) + (Ct+D)\cos(2t)$

$\rightarrow Y' = 2(At+B)\cos(2t) + A\sin(2t) + -2(Ct+D)\cos(2t) + C\cos(2t)$

$\rightarrow Y'' = -4(At+B)\sin(2t) + 2A\cos(2t) + 2A\cos(2t) + 4(Ct+D)\sin(2t)$
$\qquad + -2C\cos(2t) - C\sin(2t)$

Plug in: $Y'' + Y = 3\sin(2t) + t\cos(2t)$

① $\Rightarrow \sin(2t)\left[(At+B) - 4(At+B) + 4(Ct+D) - C\right] = 3\sin(2t)$

② $\Rightarrow \cos(2t)\left[(Ct+D) + 2A + 2A - 2C\right] = t\cos(2t)$

① $\Rightarrow A\cancel{t} - 4A\cancel{t} + 4C\cancel{t} = 0$ & $B - 4B + 4D - C = 3$

$\star$ $-3A\cancel{t} + 4C\cancel{t} = 0$ & $-3B - C + 4D = 3 \rightarrow -3B - 1 + 4D = 3$

$-3B + 4D = 4.$

$-3B + 4\left(\frac{-10}{3}\right) = 4$

$-3B = 4 + \frac{40}{3} = \frac{52}{3}$

$\Rightarrow B = \frac{52}{-3\cdot 3} = \boxed{\frac{-52}{9}}$

② $\Rightarrow \boxed{C = 1}$ & $D + 2A + 2A - 2C = 0$

$4A - 2C + D = 0$

$\downarrow$

$4\left(\frac{4}{3}\right) - 2(1) + D = 0$

$\frac{16}{3} - 2 = -D \Rightarrow \boxed{D = \frac{-10}{3}}$

$\star \Rightarrow -3A + 4 = 0$

$\Rightarrow \boxed{A = \frac{4}{3}}$

Ex: $y'' + 9y = t^2 e^{3t} + 6$

§3.5 # 7

Homogeneous: $y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = \pm 3i$

$$c_1 \cos(3t) + c_2 \sin(3t).$$

Non-homogeneous: $y'' + 9y = \underbrace{t^2 e^{3t}}_{①} + \underbrace{6}_{②}$

Split: ① $y'' + 9y = t^2 e^{3t} \longrightarrow$ Guess: $Y = (At^2 + Bt + C)e^{3t}$

② $y'' + 9y = 6 \longrightarrow$ Guess: $Y = A$.

②: $Y = A \to Y' = 0 \to Y'' = 0$

$\hookrightarrow 0 + 9A = 6 \Rightarrow A = \frac{2}{3}$. $\boxed{Y = \frac{2}{3}.}$

①: $Y = (At^2 + Bt + C)e^{3t} \Rightarrow Y' = 3(At^2 + Bt + C)e^{3t} + (2At + B)e^{3t}$

$= e^{3t}[3At^2 + (3B + 2A)t + 3C + B]$

$\Rightarrow Y'' = e^{3t}[6At + 3B + 2A] + e^{3t} \cdot 3[3At^2 + (3B + 2A)t + 3C + B]$

$\hookrightarrow e^{3t}[6At + 3B + 2A + 9At^2 + (9B + 6A)t + 9C + 3B] + e^{3t}(At^2 + Bt + C) = t^2 e^{3t}$

$\Rightarrow$ (i) $9A + A = 1 \Rightarrow A = \frac{1}{10}$

(ii) $6A + 6A + B = 0 \Rightarrow 12\left(\frac{1}{10}\right) + B = 0 \Rightarrow B = \frac{-12}{10}$

(iii) $3B + 2A + 9B + 9C + 3B + C = 0 \Rightarrow 2A + 15B + 10C = 0$

$\Rightarrow 2\left(\frac{1}{10}\right) + 15\left(\frac{-12}{10}\right) + 10C = 0$

Ex: $y'' - 3y' - 4y = 2e^{-x}$.

- Want to guess: $Y(x) = Ae^{-x}$ but we can't! (see pg 5)

  Homogeneous ODE : $y'' - 3y' - 4y = 0 \Rightarrow r^2 - 3r - 4 = 0$
  $\Rightarrow (r-4)(r+1) = 0 \Rightarrow r = 4, r = -1$

  $\hookrightarrow c_1 e^{4x} + c_2 e^{-x}$

- New guess: $Y(x) = Axe^{-x}$ ($x \cdot$ old guess)

  $Y'(x) = -Axe^{-x} + Ae^{-x} \Rightarrow Y''(x) = Axe^{-x} - Ae^{-x} - Ae^{-x}$
  $= Axe^{-x} - 2Ae^{-x}$

plug in:
$\Rightarrow (Axe^{-x} - 2Ae^{-x}) - 3(-Axe^{-x} + Ae^{-x}) - 4(Axe^{-x}) = 2e^{-x}$

$\Rightarrow (A + 3A - 4A)xe^{-x} + (-2A - 3A)e^{-x} = 2e^{-x}$

$\Rightarrow -5A = 2 \Rightarrow A = \frac{-2}{5}$

- $Y(x) = \frac{-2}{5}xe^{-x}$

Gen soln: $y = c_1 e^{4x} + c_2 e^{-x} - \frac{2}{5}xe^{-x}$.

Ex: $y'' + 2y' + y = 2e^{-x}$

Note: Homogeneous : $y'' + 2y' + y = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)(r+1) = 0 \Rightarrow r = -1 \quad r = -1$

$\Rightarrow c_1 e^{-x} + c_2 x e^{-x}$

- Want to guess: (i) $Y = Ae^{-x}$ Can't! (ii) $Y = Axe^{-x}$ Can't.

- New guess: $Y = Ax^2 e^{-x}$

  $\hookrightarrow$ HW: Show that $A = 1$!