Solve each of the following IVPs

① $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$

$$r^2 + 4r + 5 = 0$$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

Gen Sol'n:

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x)$$

IVP:

- $\boxed{1 = c_1}$
- $y' = e^{-2x}(-c_1 \sin x + c_2 \cos x) + -2e^{-2x}(c_1 \cos x + c_2 \sin x)$

$$\Rightarrow 0 = 1(c_2) - 2(c_1) = c_2 - 2 \Rightarrow \boxed{c_2 = 2}$$

$\Rightarrow$ part. sol'n

$$y = e^{-2x}(\cos x + 2 \sin x)$$

② $6y'' - 5y' + y = 0$, $y(0) = 4$, $y'(0) = 0$.

$$6r^2 - 3r - 2r + 1 = 0$$

$$\Rightarrow 3r(2r-1) - 1(2r-1) = 0$$

$$\Rightarrow (3r-1)(2r-1) = 0$$

$$\Rightarrow r = 1/3 \quad r = 1/2$$

Gen Soln:

$$y = c_1 e^{1/3 x} + c_2 e^{1/2 x}$$

IVP:

- $4 = c_1 + c_2 \rightsquigarrow c_2 = 4 - c_1$
- $y' = \frac{1}{3} c_1 e^{1/3 x} + \frac{1}{2} c_2 e^{1/2 x}$

$$\Rightarrow 0 = \frac{1}{3} c_1 + \frac{1}{2} c_2$$

$$\Rightarrow 0 = 2c_1 + 3c_2 = 2c_1 + 3(4 - c_1) = 2c_1 - 3c_1 + 12 \Rightarrow \boxed{c_1 = +12} \quad \boxed{c_2 = -8}$$

part. soln

$$y = 12e^{1/3 x} - 8e^{1/2 x}.$$

§3.4 - Repeated Roots

Ex: $y'' - 6y' + 9y = 0$

$\hookrightarrow r^2 - 6r + 9 = 0 \Rightarrow (r-3)(r-3) = 0$

$\Rightarrow r = 3,\ r = 3.$ (one root, repeated).

Know: one solution is $y_1 = \boxed{e^{3x}}$ (and constant multiples thereof)

Want: Another solution which isn't a constant multiple of e^{3x}.

Suspect: There may be a function $f(x)$ such that $y_2 = f(x)e^{3x}$ is a solution.

$\hookrightarrow y' = 3f(x)e^{3x} + f'(x)e^{3x}$

$y'' = 9f(x)e^{3x} + \underbrace{3f'(x)e^{3x} + 3f'(x)e^{3x}}_{} + f''(x)e^{3x}$

Plug in to ODE:

!!

$0 = y'' - 6y' + 9y = [9f(x)e^{3x} + \boxed{6\cancel{f'(x)e^{3x}}} + \boxed{f''(x)e^{3x}}]$

$-6[3f(x)e^{3x} + \cancel{f'(x)e^{3x}}]$

$+ 9[f(x)e^{3x}]$

$(9 - 18 + 9)f(x)e^{3x} = 0$

This part matches w/ original solution.

$\Rightarrow 0 = f''(x)e^{3x} \Rightarrow f''(x) = 0 \Rightarrow f'(x) = c_1 \Rightarrow f(x) = c_1 x + c_2.$

So: $f(x)e^{3x} = (c_1 x + c_2)e^{3x} = c_1 xe^{3x} + \boxed{c_2 e^{3x}}$ a solution!

Now, we can check the general case:

- Repeated real roots to $ar^2+br+c \iff b^2-4ac=0$

$$\iff r_1, r_2 = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$$

- If we "guess" that

$f(x)e^{(-b/2a)x}$ is a solution, we'll get

$$f''(x)=0 \iff f'(x)=c_1 \iff f(x)=c_1x+c_2 .$$

So:

<u>General Solution</u>: $y=c_1e^{(-b/2a)x}+c_2xe^{(-b/2a)x}$.

<u>Ex</u>: Solve the IVP: $y''-y'+0.25y=0$, $y(0)=2$, $y'(0)=1/3$.

<u>Char Eq</u>: $r^2-r+0.25=0$

$$\Rightarrow r=\frac{+1\pm\sqrt{1-4(1)(0.25)}}{2}=\frac{1}{2} \text{ (repeated)}.$$

<u>Gen Sol'n</u>: $y=c_1e^{1/2x}+c_2xe^{1/2x}$

<u>IVP</u>: • $y(0)=2 \iff 2=c_1$

• $y'(x)=\frac{1}{2}c_1e^{1/2x}+\frac{1}{2}c_2xe^{1/2x}+c_2e^{1/2x}$

& $y'(0)=\frac{1}{3} \iff \frac{1}{3}=\frac{1}{2}c_1+c_2 = 1+c_2 \Rightarrow c_2=\frac{-2}{3}$.

<u>Part. Sol'n</u>: $y=2e^{1/2x}-\frac{2}{3}xe^{1/2x}$.