§3.3 - Complex roots of Charcteristic Eq

Recall: If $ay''+by'+cy=0$ is a 2nd order linear homogeneous ODE w/ constant coefficients, its characteristic eq. $ar^2+br+c=0$ has 3 cases:

① $b^2-4ac>0 \rightsquigarrow$ 2 real solutions (§3.1)

② $b^2-4ac=0$

③ $b^2-4ac<0 \rightsquigarrow$ 2 complex solutions (THIS section!)

- Suppose we're in this case. Then $r_1=\lambda+\mu i$ & $r_2=\lambda-\mu i$ for some $\lambda,\mu\in\mathbb{R}$.
- Expect: General solution

$$\begin{aligned} y &= c_1e^{r_1x}+c_2e^{r_2x} \\ &= c_1e^{(\lambda+\mu i)x}+c_2e^{(\lambda-\mu i)x} \\ &= e^{\lambda}\left(c_1e^{\mu ix}+c_2e^{-\mu ix}\right). \quad (*) \end{aligned}$$

But what does this mean?! what is $e^{(\text{imaginary }\#)}$?

Fact (Euler's Identity)

$$e^{ix}=\cos x+i\cdot\sin(x)$$ (Be able to show this using MacLaurin series!)

$\Rightarrow e^{-ix}=\cos x-i\sin x$

Plug in to $(*)$: Expected general solution is

$$y=e^{\lambda}\left(c_1e^{i\mu x}+c_2e^{i(-\mu x)}\right)=e^{\lambda}\left(c_1(\cos(\mu x)+i\sin(\mu x))+c_2(\cos(\mu x)-i\sin(\mu x))\right)$$

Ex: $y'' + y' + 9.25y = 0$

↳ Char Eq: $r^2 + r + 9.25 = 0$

$$\Rightarrow r = \frac{-1 \pm \sqrt{1 - 4(1)(9.25)}}{2} = \frac{-1 \pm \sqrt{1 - 37}}{2}$$

$$= \frac{-1 \pm \sqrt{-36}}{2} = \frac{-1 \pm 6i}{2} = \frac{-1}{2} \pm 3i.$$

Expected general solution:

$$y = c_1 \overset{y_1}{\boxed{e^{(\frac{-1}{2}+3i)x}}} + c_2 \overset{y_2}{\boxed{e^{(\frac{-1}{2}-3i)x}}}$$

$$(*) \quad = e^{\frac{-1}{2}x}\Big[c_1\big(\cos(3x) + i\sin(3x)\big) + c_2\big(\cos(3x) - i\sin(3x)\big)\Big]$$

We don't like this because real-valued ODE w/ real coefficients "should" have real solutions!

Theorem: (3.2.6) Consider the ODE $y'' + p(x)y' + q(x)y = 0$ w/ p, q continuous. If

$$y = u(x) + iv(x)$$

is a complex-valued solution, then $u(x)$ & $v(x)$ are (real-valued) solutions.

$\Rightarrow$ In above example, we can collect real & imaginary parts:

- In y_1: $\exp(\frac{-1}{2}+3i)x = e^{-1/2x}(\cos(3x) + i\sin(3x))$ solution

  $\Rightarrow e^{-1/2x}\cos(3x)$ & $e^{-1/2x}\sin(3x)$ solutions.

- Same from y_2!

So:

Ex: $y'' + y' + 9.25y = 0$

$$\Downarrow$$

$$y_1 = e^{-\frac{1}{2}x}(\cos(3x) + i\sin(3x))$$
$$y_2 = e^{-\frac{1}{2}x}(\cos(3x) - i\sin(3x))$$

both solutions

$$\Downarrow$$

$e^{-\frac{1}{2}x}\cos(3x)$ & $e^{-\frac{1}{2}x}\sin(3x)$ both solutions

So, the real general solution is:

$$y = c_1 e^{-\frac{1}{2}x}\cos(3x) + c_2 e^{-\frac{1}{2}x}\sin(3x).$$

Ex: (i) $16y'' - 8y' + 145y = 0 \xrightarrow{\text{char eq.}} 16r^2 - 8r + 145 = 0$

$$\Longrightarrow r = \frac{1}{4} \pm 3i.$$

General solution: $y = c_1 e^{\frac{1}{4}x}\cos(3x) + c_2 e^{\frac{1}{4}x}\sin(3x).$

(ii) Solve the IVP $16y'' - 8y' + 145y = 0$, $y(0) = -2$, $y'(0) = 1$.

Gen soln: $y = c_1 e^{\frac{1}{4}x}\cos(3x) + c_2 e^{\frac{1}{4}x}\sin(3x) = e^{\frac{1}{4}x}(c_1\cos(3x) + c_2\sin(3x))$

$\Rightarrow$ (a) $\boxed{-2 = c_1 + 0}$ & (b) $y' = e^{\frac{1}{4}x}(-3c_1\sin(3x) + 3c_2\cos(3x)) + \frac{1}{4}e^{\frac{1}{4}x}(c_1\cos(3x) + c_2\sin(3x))$

$\Rightarrow 1 = 1(0 + 3c_2) + \frac{1}{4}(c_1 + 0)$

$\Rightarrow 1 = 3c_2 + \frac{1}{4}(-2) \Rightarrow \boxed{c_2 = \frac{1}{2}.}$

Part. Soln:

$y = e^{\frac{1}{4}x}\left(-2\cos(3x) + \frac{1}{2}\sin(3x)\right)$