§ 3.2 (Cont'd)

Recall: • The first-order linear IVP $y' + p(x)y = q(x)$, $y(x_0) = y_0$ has a unique solution on the interval I in which both p, q are continuous. & which contains x_0.

• A second order linear ODE has the general form

$$y'' + p(x)y' + q(x)y = g(x).$$

• The Wronskian of y_1 & y_2 is the function

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Goal: ① State an equiv. to above thm for second order linear ODE.

② Discuss more properties of the Wronskian.

Thm 3.2.1 (Existence & Uniqueness Thm for 2nd order Linear IVP)

Consider the IVP $y'' + p(x)y' + q(x)y = g(x)$, $y(x_0) = y_0$, $y'(x_0) = y_0'$, where p, q, & g are continuous on an open interval I containing x_0. Then this IVP has a unique solution y. & this y exists throughout I.

Ex: Find the longest interval in which a unique solution to $(x^2-3x)y''+xy'-(x+3)y=0$, $y(1)=2$, $y'(1)=1$. is certain to exist.

↳ Rewrite: $y''+\underbrace{\frac{x}{x(x-3)}}_{p}y'-\underbrace{\frac{x+3}{x(x-3)}}_{q}y=\underbrace{0}_{g}$ $y(1)=2$ $y'(1)=1$

- g cont. everywhere
- p cont. for $x\neq 0, x\neq 3$: $(-\infty,0)\cup(0,3)\cup(3,\infty)$
- q cont for $x\neq 0, x\neq 3$: $(-\infty,0)\cup(0,3)\cup(3,\infty)$
- $x_0=1 \Rightarrow x_0$ in this interval.

By thm, longest interval is $(0,3)$.

Ex: Same directions as ↑ for IVP

$\boxed{\#12, §3.2}$

$(x-2)y''+y'+((x-2)\tan x)y=0$ $y(3)=1$, $y'(3)=2$

Rewrite: $y''+\underbrace{\frac{1}{x-2}}_{p}y'+\underbrace{\frac{(x-2)\tan x}{x-2}}_{q}y=\underbrace{0}_{g}$

$\boxed{\text{Ans: } (2, \frac{3\pi}{2})}$

- g cont. everywhere
- p cont @ $x\neq 2$: $(-\infty,2)\cup(2,\infty)$
- q cont @ $x\neq 2$, $x\neq \frac{n\pi}{2}$ for odd integers n:

  $\ldots\cup(-3\pi/2,-\frac{\pi}{2})\cup(-\frac{\pi}{2},\frac{\pi}{2})\cup(\frac{\pi}{2},2)\cup(2,\frac{3\pi}{2})\cup\ldots$
- $x_0=3$ in this interval

• Now, we shift to a general formula for finding Wronskians.

Abel's Theorem

If y_1, y_2 are solutions to the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

then

$$W(y_1, y_2) = C \cdot \exp\left[-\int p(x)\,dx\right]$$

where C is a constant depending on y_1 & y_2.

Ex: Consider ODE

$\boxed{\S 3.2 \# 32}$ $$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \quad \alpha = \text{const.}$$

Find the Wronskian of two solutions to the ODE w/o solving it.

Rewrite: $$y'' \underbrace{- \frac{2x}{1-x^2}}_{p(x)} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$$

By thm: $$W(y_1, y_2) = C\exp\left(-\int \frac{-2x}{1-x^2}\,dx\right)$$

$u = 1-x^2$
$du = -2x\,dx$

$$= C\exp\left(-\ln|1-x^2|\right)$$

$$= C \cdot |1-x^2|^{-1}$$

$$= C_2(1-x^2)^{-1}$$ where $C_2 = \pm C$ dep. on whether $|\cdots|$ quant. has $\ldots > 0$ or $\ldots < 0$

$$= \frac{C_2}{1-x^2}.$$

Ex: If f, g, h are differentiable, then what is $w(fg, fh)$?

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$$w(fg, fh) = \det\begin{pmatrix} fg & fh \\ fg' + gf' & fh' + hf' \end{pmatrix}$$

$$= fg(fh' + hf') - fh(fg' + gf')$$

$$= f^2gh' + \cancel{fghf'} - f^2hg' \cancel{- fghf'}$$

$$= f^2(gh' - hg')$$

$$= f^2 \det\begin{pmatrix} g & h \\ g' & h' \end{pmatrix}$$

$$= f^2\, w(g, h).$$

Ex: $[p(x)y']' + q(x)y = 0$

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$$\Rightarrow p(x)y'' + p'(x)y' + q(x)y = 0$$

$$\Rightarrow w(y_1, y_2) = c\exp\left(-\int \frac{p'(x)}{p(x)}\,dx\right) \qquad u = p(x), \quad du = p'(x)dx$$

$$= c\exp\left(-\int \frac{1}{u}\,du\right)$$

$$= c\cdot\exp(\ln|u|^{-1})$$

$$= \frac{c}{|p(x)|} = \frac{c_2}{p(x)} \quad \text{where } c_2 \text{ const.}$$