§3.1 – Homogeneous Eq's w/ Constant Coefficients

In this chapter, we study 2nd order ODEs:

↳ ∘ Equations involving x, $y=y(x)$, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$.

∘ In general, we write a 2nd order ODE as

$$y'' = f(x, y, y') \quad \text{for some } f.$$

Defs: ① 2nd order ODE is linear if it has the form

compare w/ first order linear: $y' + p(x)y = q(x)$!

$$y'' + p(x)y' + q(x)y = g(x)$$

where p, q, g have only x's & constants!

[More general: $A(x)y'' + B(x)y' + C(x)y = H(x)$].

② If an ODE isn't linear, it's nonlinear.

③ A second order linear ODE is homogeneous if $g(x)=0$, i.e. if it has the form

$$y'' + p(x)y' + q(x)y = 0.$$

④ If an ODE isn't homogeneous, it's called nonhomogeneous.

In this section, we study 2nd order linear equations which are homogeneous, and we restrict attention to the ones of the form

$$A(x)y'' + B(x)y' + C(x)y = 0, \quad \text{where } A(x), B(x), C(x) = \text{constant}.$$

(ODEs w/ nonconstant coefficients are hard)

⑤ A 2nd order IVP consists of an ODE which is 2nd order along with two initial conditions: $y(x_0) = y_0$ and $y'(x_0) = y_0'$.

<u>Ex:</u> Solve the equation $y''-y=0$. Also, solve the IVP given $y(0)=2$ & $y'(0)=-1$.

↳ <u>Observe:</u> (i) $y''-y=0 \iff y''=y$. One obvious solution is $y=e^x$.

(ii) If $y=e^x$ is a solution, so is $y=c_1e^x$ for all const c_1: $y=c_1e^x \Rightarrow y'=c_1e^x \Rightarrow y''=c_1e^x$.

(iii) Another solution is $y=e^{-x}$: $y'=-e^{-x} \Rightarrow y''=e^{-x}$.

(iv) As in (ii), so is $y=c_2e^{-x}$ for all const c_2

(v) If y_1 & y_2 are solutions, so is y_1+y_2:

$$
\begin{aligned}
(y_1+y_2)''-(y_1+y_2) &= y_1''+y_2''-y_1-y_2 \\
&= (y_1''-y_1)+(y_2''-y_2) \\
&= 0+0=0.
\end{aligned}
$$

↳ Combining (i)–(v): The function $y=c_1e^x+c_2e^{-x}$ is a solution. This is actually the general solution of the ODE but we don't know that yet!

↳ For IVP:

- Start w/ $y=c_1e^x+c_2e^{-x}$ & $y(0)=2$:

$$\boxed{2=c_1+c_2} \quad (*)$$

- Now, find y' and use $y'(0)=-1$:

$$y'=c_1e^x-c_2e^{-x} \text{ \& } y'(0)=-1 \Rightarrow \boxed{-1=c_1-c_2.} \quad (**)$$

- Add $(*)$ & $(**)$: $1=2c_1 \Rightarrow \boxed{c_1=1/2}$

- Plug in either $(*)$ or $(**)$: $2=\frac{1}{2}+c_2 \Rightarrow \boxed{c_2=\frac{3}{2}}$

- Particular (Not EXACT!!) soln: $\boxed{y=\frac{1}{2}e^x+\frac{3}{2}e^{-x}.}$

To generalize:

- start w/ ODE $\boxed{ay'' + by' + cy = 0}$ (*) for const's a, b, c.
- Look for a solution $y = e^{rt}$ where r = const <u>TO BE DETERMINED</u>!
- Plug in:

$$y = e^{rt} \Rightarrow y' = re^{rt} \Rightarrow y'' = r^2 e^{rt}, \quad \text{so}$$

$$ay'' + by' + cy = 0 \iff ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$
$$\iff e^{rt}(ar^2 + br + c) = 0$$
$$\iff \boxed{ar^2 + br + c = 0}!$$

DEF: This is the <u>characteristic eq.</u> of the ODE!

- Solutions of the ODE (*) having the form $y = e^{rt}$ correspond to roots of the characteristic equation $ar^2 + br + c = 0$.

↳ Solve for r: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Downarrow$

There are three cases!

$\boxed{\text{(i) } b^2 - 4ac > 0 \Rightarrow \text{two distinct real roots } r_1 \ \& \ r_2.}$

(ii) $b^2 - 4ac = 0 \Rightarrow$ one "repeated" real root r_1

(iii) $b^2 - 4ac < 0 \Rightarrow$ two distinct non-real roots!

This is the case we care about in <u>this</u> section!

General Solution

$ay''+by'+cy=0$

Given a 2^{nd} order ODE w/ characteristic equation $ar^2+br+c=0$ having real roots $r_1 \neq r_2$, the general solution is

$$y_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Ex: ① $y''+5y'+6y=0 \rightsquigarrow$ char eq: $r^2+5r+6=0$

$\Rightarrow (r+2)(r+3)=0$

$\Rightarrow r=-2$ & $r=-3$.

These are distinct, so general solution:

$$y = c_1 e^{-2t} + c_2 e^{-3t}.$$

CHECK

Is it a solution?

$y = c_1 e^{-2t} + c_2 e^{-3t} \Rightarrow y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} \Rightarrow y'' = 4c_1 e^{-2t} + 9c_2 e^{-3t}$,

so: $y''+5y'+6y = (4c_1 e^{-2t} + 9c_2 e^{-3t}) + 5(-2c_1 e^{-2t} - 3c_1 e^{-3t}) + 6(c_1 e^{-2t} + c_2 e^{-3t})$

$= c_1 e^{-2t}(4-10+6) + c_2 e^{-3t}(9-15+6)$

$= 0!$

② IVP: $y''+5y'+6y=0$, $y(0)=2$, $y'(0)=3$.

$2 = c_1 + c_2$ & $3 = -2c_1 - 3c_2 \rightsquigarrow$ solve!

Ex: (a) Find the solution of the IVP $4y''-8y'+3y=0$, $y(0)=2$, $y'(0)=\frac{1}{2}$.

(b) Find the max of the solution.

(a) Char Eq: $4r^2-8r+3=0$

$\Rightarrow 4r^2-2r-6r+3=0 \Rightarrow 2r(2r-1)-3(2r-1)=0$

$\Rightarrow (2r-3)(2r-1)=0$

$\Rightarrow r=\frac{3}{2} \quad r=\frac{1}{2}$

So gen soln: $\boxed{y=c_1e^{3/2x}+c_2e^{1/2x}}$.

↳ IVP:
- $2=c_1+c_2$
- $y'=3/2\,c_1e^{3/2x}+\frac{1}{2}c_2e^{1/2x} \Rightarrow \frac{1}{2}=\frac{3}{2}c_1+\frac{1}{2}c_2$

$\Rightarrow 1=3c_1+c_2$.

- Subtract: $1=-2c_1 \Rightarrow c_1=-\frac{1}{2}$

$\Rightarrow -\frac{1}{2}+c_2=2 \Rightarrow c_2=5/2$

$\Rightarrow$ Particular soln: $y=\frac{-1}{2}e^{3/2x}+\frac{5}{2}e^{1/2x}$.

(b) This doesn't have a max: $\lim\limits_{x\to\infty} y=\infty$!

↳ • If we had $y=9e^{-2x}-7e^{-3x}$, for example, it does have a max: We know $\lim\limits_{x\to\infty} y=0$, and $y'=-18e^{-2x}+21e^{-3x}$ is positive at $x=0 \Rightarrow$ initially increasing!

- To find max, set $y'=0$: $-18e^{-2x}+21e^{-3x}=0 \Rightarrow -18e^{x}+21=0$

$\Rightarrow e^x=\frac{21}{18} \Rightarrow \boxed{\begin{aligned}&x=\ln\left(\tfrac{7}{6}\right)\\&y=9e^{-2\ln(7/6)}-7e^{-3\ln(7/6)}\end{aligned}}$ where max occurs.