§2.8 – The Existence & Uniqueness Theorem

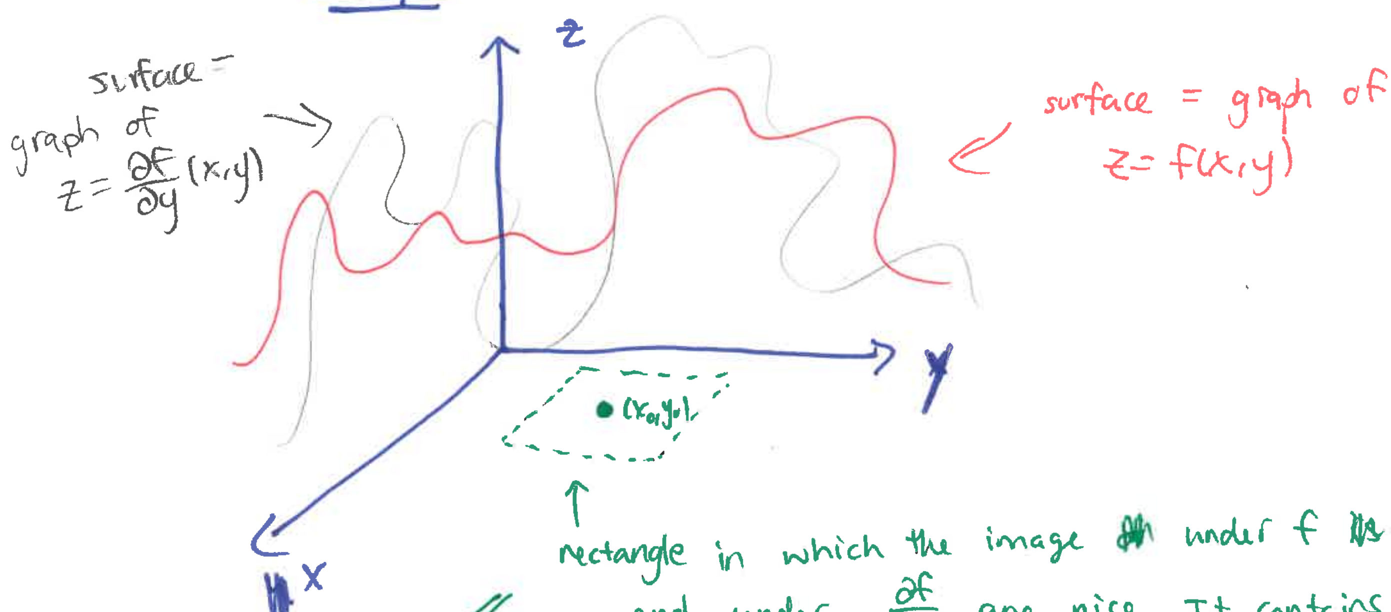
Recall: There's a theorem about the existence/uniqueness of solutions for linear first-order ODEs. We want something more general.

Theorem 2.4.2 (aka 2.8.1) ⟶ "The Existence & Uniqueness Theorem"

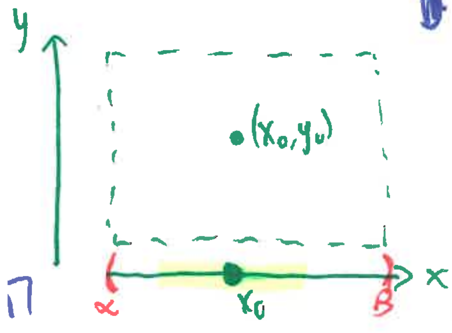
Let $\frac{dy}{dx} = f(x,y)$ be a first-order ODE. & consider the IVP

$$\frac{dy}{dx} = f(x,y), \qquad y(x_0) = y_0. \qquad \alpha < x < \beta, \ \gamma < y < \delta$$

If f & $\frac{\partial f}{\partial y}$ are continuous in some rectangle containing the pt (x_0, y_0), then in some x-interval contained in $\alpha < x < \beta$ containing x_0, there is a <u>unique</u> solution to the ODE.



rectangle in which the image under f and under $\frac{\partial f}{\partial y}$ are nice. It contains (x_0, y_0)



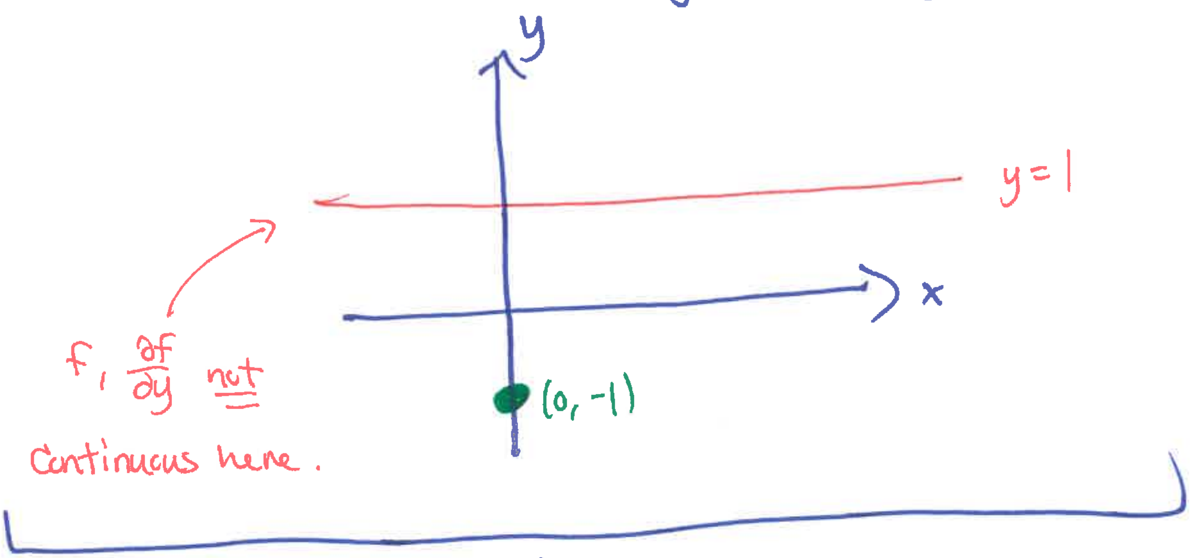
= some x-interval contained in $\alpha < x < \beta$ containing x_0 on which IVP soln exists & is unique.

Ex: $\dfrac{dy}{dx} = \dfrac{3x^2+4x+2}{2(y-1)}$. $y(0) = -1$.

$\Rightarrow$ $f(x,y) = \dfrac{3x^2+4x+2}{2(y-1)}$ & $\dfrac{\partial f}{\partial y} = -\dfrac{3x^2+4x+2}{2(y-1)^2}$

(use quotient rule)

$\to$ Both are continuous everywhere **except** when $y=1$!



$\Downarrow$

There exists **some** rectangle (lots of them, actually) around $(0,-1)$ s.t. f, $\dfrac{\partial f}{\partial y}$ continuous in that rectangle.

$\hookrightarrow$ By existence & uniqueness theorem, this IVP has a unique solution in **some** interval about $x=0$.

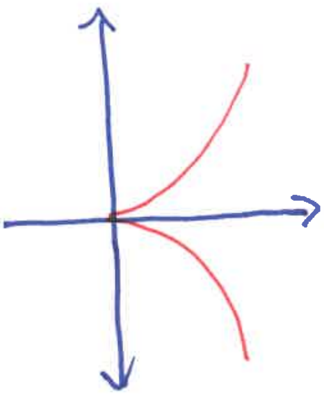
(Don't know which interval, though.
By other methods, can show $x > -2$).

Note:

- If f continuous but $\frac{\partial f}{\partial y}$ isn't, then a solution exists but may not be unique.

  ↳ Ex: $\frac{dy}{dx} = y^{1/3}$, $y(0)=0$.

  Here, $f(x,y) = y^{1/3}$ is continuous everywhere, but $\frac{\partial f}{\partial y}$ does not exist at $x=0$.



$\Downarrow$

There is a solution but it's not unique!

↳ $y_1 = \left(\frac{2}{3}x\right)^{3/2}$

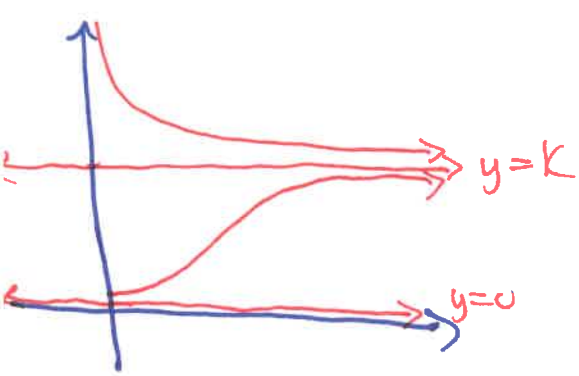
$y_2 = -\left(\frac{2}{3}x\right)^{3/2}$

$y_3 = 0$

Check that these are all solutions!

- By Existence & uniqueness theorem, the graphs of two solutions cannot intersect when f, $\frac{\partial f}{\partial y}$ both continuous.

  ↳ Ex: In §2.5, we studied $\frac{dy}{dx} = r\left(1-\frac{y}{K}\right)y$.



Even though the solutions in the y-interval $(0,K)$ (for example) limit towards $y=K$, the never intersect it!

Ex. (§2.4 # 11) State where in the xy-plane the hypotheses of Thm 2.4.2 are satisfied:

$$\frac{dy}{dx} = \frac{1+x^2}{3y-y^2}$$

$$\underbrace{f(x,y) = \frac{1+x^2}{3y-y^2}}_{\text{continuous when}}$$

$3y-y^2 \neq 0$

$\iff y(3-y) \neq 0$

$\iff y \neq 0$ and $y \neq 3$.

$$\frac{\partial f}{\partial y}(x,y) \underset{\text{rule}}{\overset{\text{quotient}}{=}} \frac{0-(1+x^2)(3-2y)}{(3y-y^2)^2}$$

$$= \underbrace{-\frac{(1+x^2)(3-2y)}{(3y-y^2)^2}}_{\text{continuous when}}$$

$3y-y^2 \neq 0 \ldots$

So: Hypotheses of thm valid when

$y \neq 0$ & $y \neq 3$!

Note: This theorem is a generalization of the analogous thm for linear ODEs: If $y' + p(x)y = q(x)$ is ODE, then

$$y' = \underbrace{q(x) - p(x)y}_{f(x,y)} \Rightarrow \frac{\partial f}{\partial y} = -p(x).$$

So, f & $\frac{\partial f}{\partial y}$ both continuous IFF $p(x)$ and $q(x)$ both continuous!

Ex: $\boxed{\S 2.4 \# 9}$ $y' = \dfrac{\ln|xy|}{1-x^2+y^2}$

$\ln|xy| = \ln(\pm xy) \xrightarrow{\frac{\partial}{\partial y}} \dfrac{1}{\pm xy} \cdot \pm x = \dfrac{\pm x}{\pm xy} = \dfrac{1}{y}$

$$\underbrace{f(x,y) = \frac{\ln|xy|}{1-x^2+y^2}}$$

continuous when:

- $|xy| > 0$

$\Rightarrow |xy| \neq 0$

$\Rightarrow |x||y| \neq 0$

$\Rightarrow x \neq 0$ and $y \neq 0$

- $1-x^2+y^2 \neq 0$

$\Rightarrow x^2 \neq 1+y^2$

$\Rightarrow x \neq \pm\sqrt{1+y^2}$

$$\underbrace{\frac{\partial f}{\partial y} = \frac{(1-x^2+y^2)\cdot\frac{1}{y} - \ln|xy|\,(2y)}{(1-x^2+y^2)^2}}$$

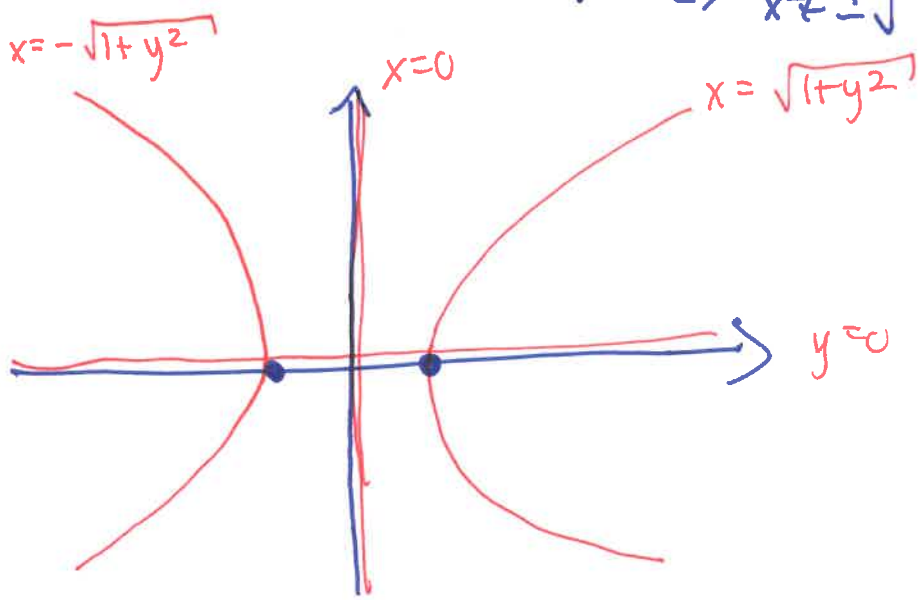
continuous when:

- $y \neq 0$
- $|xy| > 0$

$\Rightarrow x \neq 0$ & $y \neq 0$

- $1-x^2+y^2 \neq 0$

$\Rightarrow x \neq \pm\sqrt{1+y^2}$



hypotheses of thm valid everywhere else.

/ = bad parts (thm hypotheses not satisfied)