§ 2.6 - Exact Equations

Recall: The partial derivative of a 2var function $f(x,y)$ WRT x (or y) is the function $\frac{\partial f}{\partial x}$ (or $\frac{\partial f}{\partial y}$) obtained by treating y (or x) as const. & taking "normal derivative" of result.

Ex: ① $f(x,y) = 2xy + \sin(xy) + e^y + e^x$

$$\hookrightarrow \underbrace{\frac{\partial f}{\partial x}}_{f_x} = 2y + y\,\cos(xy) + e^x$$

$$\underbrace{\frac{\partial f}{\partial y}}_{f_y} = 2x + x\cos(xy) + e^y$$

=0 for linear

Ex: $\underbrace{2x+y^2}_{f_x} + \underbrace{2xy}_{f_y}y' = 0$ is not separable or linear!

↳ Note: If I let $f(x,y) = x^2 + xy^2$, then

$$\begin{cases} f_x = 2x+y^2 \\ f_y = 2xy \end{cases} \Rightarrow \text{ODE is } f_x + f_y \frac{dy}{dx} = 0$$

Now, if y is a function of x:

$$\frac{df(x,y)}{dx} = (x \text{ der of } f) + (y \text{ der. of } f)$$

$$= f_x + f_y \frac{dy}{dx}$$

fact from calc III

so ODE is

$$\frac{df}{dx} = 0 \iff \frac{d}{dx}(x^2+xy^2) = 0 \iff \boxed{x^2+xy^2 = C}$$

Def: An ODE of the form

$$M(x,y) + N(x,y)\,y' = 0$$

is <u>exact</u> iff $M_y = N_x$, i.e. $\exists$ function $f(x,y)$ such that

$$f_x = M(x,y) \quad \text{and} \quad f_y = N(x,y)$$

To solve: Find f!

Ex: $\overbrace{(y\cos x + 2xe^y)}^{M} + \overbrace{(\sin x + x^2e^y - 1)}^{N}y' = 0$ is exact b/c

$$M_y = \cos x + 2xe^y$$

$$N_x = \cos x + 2xe^y$$

$$\Rightarrow \exists\, f \text{ w/ } f_x = M = y\cos x + 2xe^y$$

$$f_y = N = \sin x + x^2e^y - 1 \qquad (*)$$

Find f:

$$f = \int f_x\,dx = \overbrace{y\sin x + x^2e^y + k(y)}^{(**)} \quad \text{for some func } k$$

Find f_y & compare w/ $(*)$:

$$f_y = \sin x + x^2e^y + k'(y) \overset{\text{by}(*)}{=\!=} \sin x + x^2e^y - 1 \Rightarrow k'(y) = -1$$

can omit.

$$\Rightarrow k(y) = y + C. \quad \text{So: } f = y\sin x + x^2e^y + y + \boxed{C} \text{ (from } (**)).$$

Ex (Cont'd)

Claim: Solution to the ODE is $f(x,y) = C$

$$\iff y\sin x + x^2 e^y + y = C.$$

To see this:

original ODE has form

$$f_x + f_y\, y' = 0$$

$$\iff \frac{d}{dx}\big(f(x,y)\big) = 0 \quad \left(\text{b/c } \frac{d}{dx}\big(f(x,y)\big) = f_x + f_y\, y' \text{ by multivar chain rule.}\right)$$

$$\iff \int \frac{d}{dx}\big(f(x,y)\big)\,dx = \int 0\, dx$$

$$\iff f(x,y) = \text{const}.$$

Ex: $\boxed{\text{§2.6 \#10}}$ $\left(\frac{y}{x} + 6x\right) + (\ln x - 2)y' = 0, \quad x > 0$

- $\underline{\text{is}}$ exact: $M_y = \frac{1}{x}$ & $N_x = \frac{1}{x}$.
- Find f:
  - $\underline{\text{Want}}$ $f_x = \overset{\frac{y}{x}+6x}{\overset{\parallel}{M}}$ and $f_y = \overset{\ln x - 2}{\overset{\parallel}{N}}$ (*)
  - $\underline{\text{If}}$ $f_x = M$, then $f_x = \frac{y}{x} + 6x \iff f = \int \frac{y}{x} + 6x\, dx$ (**)
  $$= y\ln(x) + 3x^2 + h(y).$$
  - Find $h(y)$ by taking f_y (from (**)) & comparing w/ (*):
  (**) $\Rightarrow f_y = \ln x + h'(y) \overset{\text{by (*)}}{=} \ln x - 2 \Rightarrow h'(y) = -2 \Rightarrow h(y) = -2y.$
  - Plug in to (**): $f = y\ln(x) + 3x^2 - 2y$

cont'd on next pg

Ex (Cont'd)

- use f to solve: Solution to ODE has form $f(x,y) = \text{const}$,

i.e. $\boxed{y\ln(x) + 3x^2 - 2y = C.}$

Integrating Factors

- If $\frac{M_y - N_x}{N}$ depends only on x, then the function $m(x)$ satisfying $\frac{dm}{dx} = m\left(\frac{M_y - N_x}{N}\right)$ is an integrating factor which makes the ODE exact. $m = \exp\left(\int \frac{M_y - N_x}{N}\,dx\right)$

Ex: $(3xy + y^2) + (x^2 + xy)y' = 0$ not exact, as

$$M_y = 3x + 2y \quad \& \quad N_x = 2x + y.$$

- Note: $\frac{M_y - N_x}{N} = \frac{(3x+2y) - (2x+y)}{x^2 + xy} = \frac{x+y}{x(x+y)} = \frac{1}{x}$

depends only on x.

- Find $m(x)$ s.t. $\frac{dm}{dx} = m\left(\frac{M_y - N_x}{N}\right)$:

(for this prob.) $\Rightarrow \frac{dm}{dx} = \frac{m}{x} \Rightarrow \frac{dm}{m} = \frac{dx}{x} \Rightarrow m = x.$

- Multiply ODE by m:

$\underbrace{x(3xy + y^2)}_{M = 3x^2y + xy^2} + \underbrace{x(x^2 + xy)}_{N = x^3 + x^2y}y' = 0$ is exact:

$$M_y = 3x^2 + 2xy \quad \& \quad N_x = 3x^2 + 2xy.$$

Note:

Similarly, if $\dfrac{N_x - M_y}{M}$ depends only on y, then

$$\exp\left(\int \frac{N_x - M_y}{M}\,dy\right)$$ is an integrating factor.

Note: This in M; in the other case, it's an N:

$\dfrac{N_x - M_y}{N}$ depends on x or $\dfrac{M_y - N_x}{M}$ depends on y

$$\text{I.F.} = \exp\left(\int \frac{N_x - M_y}{N}\,dx\right) \qquad \text{I.F.} = \exp\left(\int \frac{M_y - N_x}{M}\,dy\right)$$

order swapped too

N vs. M