Goals of section 2.5

- Study important class of first-order ODEs
- Using that class, develop non-slope-field tools for qualitatively predicting what ODE solutions will look like.
  - ↳ For $\frac{dy}{dx} = f(y)$, use y-versus-$f(y)$ plots to get ideas of what solutions look like in the xy-plane.

Keywords

equilibrium solution, phase line, asymptotically stable, asymptotically unstable.

§ 2.5 - Autonomous Equations

- In this section, we're going to learn about ways to study a very important class of first order ODE qualitatively.

Def: An ODE of the form $\frac{dy}{dx} = f(y)$ is said to be autonomous.

Note: ① A general first order ODE has the form $\frac{dy}{dx} = f(x,y)$, e.g. separable $\left(h(y)\,dy = g(x)\,dx \iff \frac{dy}{dx} = \frac{g(x)}{h(y)}\right)$ and linear $\left(g(x)\frac{dy}{dx} + h(x)y = k(x) \iff \frac{dy}{dx} = \frac{1}{g(x)}\left(k(x) - h(x)y\right)\right)$.

↳ Autonomous ones are special because the RHS has no independent variable!

② Every autonomous ODE is separable: $\frac{dy}{dx} = f(y) \iff \frac{dy}{f(y)} = dx$.

Ex: (i) Exponential Growth

If $\frac{dy}{dx}$ is proportional to y, then $\frac{dy}{dx} = ry$ for some y. This is autonomous, and solving yields ↳ $f(y) = ry$

$$\frac{dy}{dx} = ry \iff \frac{dy}{y} = r\,dx \iff \int \frac{dy}{y} = \int r\,dx \iff \ln|y| = rx + C,$$

and for $y > 0$, $\ln(y) = rx + C \iff y = e^{rx+C} = e^{rx}e^{C} = Ce^{rx}$.

This is the formula for exponential growth. (r = growth or decay rate)

↑

Because this model always gets bigger or smaller, it has no "dynamical" properties: The slope fields are "boring" and there isn't much worth studying qualitatively. →

(ii) Logistic growth

Given a positive constant a, consider the ODE

$$\frac{dy}{dx} = (r-ay)y. \qquad \left[f(y) = (r-ay)y \right]$$

$$= r\left(1-\frac{a}{r}y\right)y$$

$$= r\left(1-\frac{y}{r/a}\right)y$$

$$= r\left(1-\frac{y}{K}\right)y \quad \text{for } K = r/a.$$

This represents a model where the rate of growth depends on the population:

- For y small, $f(y) \approx ry \rightsquigarrow$ similar to exponential!
- As $y \nearrow$, $(r-ay)$ decreases; and
- $r-ay < 0$ when $y > r/a (= K)$.

So,
- r is the "intrinsic growth rate" (rate in absence of any limitations)
- K = "capacity" (point past which growth becomes decay).

The fact that this change occurs means this model has interesting dynamics, and as such, it's a good example for studying things qualitatively

↳ Throughout, we consider $\frac{dy}{dx} = f(y)$ for $f(y) = r\left(1-\frac{y}{K}\right)y$!

Equilibrium Solutions

↳ An equilibrium solution to an ODE is a constant solution, i.e. a solution y to the ODE $\frac{dy}{dx} = 0$. (aka critical pts; also, $\frac{dy}{dx} = f(y)$ & $\frac{dy}{dx} = 0 \Rightarrow f(y) = 0 \Rightarrow$ these are just roots of f)

- For the logistic model,

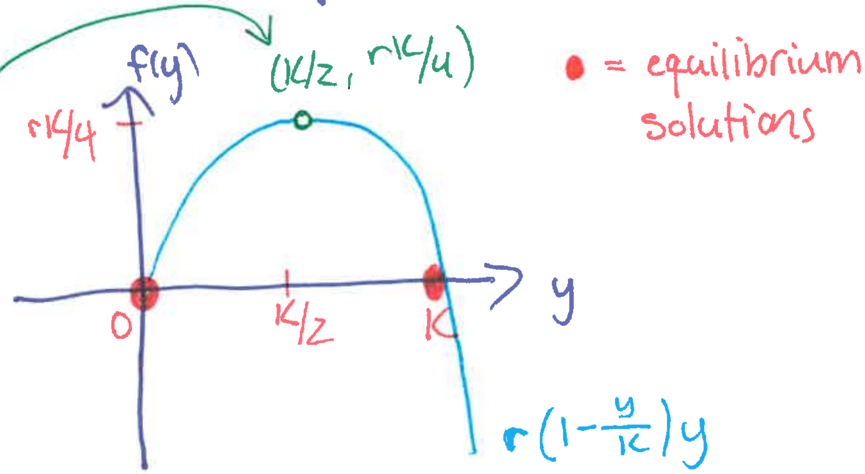
$$\frac{dy}{dx} = 0 \iff r\left(1-\frac{y}{K}\right)y = 0 \iff \begin{cases} \text{(i) } r=0, \longrightarrow \text{impossible (choose } r\neq 0 \\ \text{(ii) } y=0, \text{ or} \\ \text{(iii) } 1-\frac{y}{K} = 0 \rightsquigarrow 1 = \frac{y}{K} \Rightarrow y = K \end{cases}$$

So, equilibrium points are $\boxed{y=0 \ \& \ y=K.}$

The goal

To visualize solutions of ODE (e.g. $\frac{dy}{dx} = r\left(1-\frac{y}{K}\right)y$) fast w/o solving it. To do this, we're going to graph y v.s. $f(y)$ $\left(= r\left(1-\frac{y}{K}\right)y\right.$ for our example$\left.\right)$ and fill in some measurements:

what we know now.



B/c $f(y) = r\left(1-\frac{y}{K}\right)y$ is a parabola, we can fill in more:

- $r\left(1-\frac{y}{K}\right)y \equiv \underbrace{ry}_{b} - \underbrace{\boxed{\frac{ry^2}{K}}}_{a}$, so vertex is at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$$= \left(\frac{-r}{2(-r/K)}, f\left(\frac{-r}{2(-r/K)}\right)\right) = \left(\frac{K}{2}, f\left(\frac{K}{2}\right)\right) = \left(\frac{K}{2}, r\left(1-\frac{K/2}{K}\right)\frac{K}{2}\right)$$

$$= \boxed{\left(K/2, rK/4\right).}$$

Filling in the y-axis

- Currently, our y-axis is partitioned:

- we want to know what $f(y)$ (i.e. $\frac{dy}{dx}$) tells us about y in the subintervals ①, ②, ③.

Recall: $y = y(x)$ is a function; we don't know what it does w/o studying it!

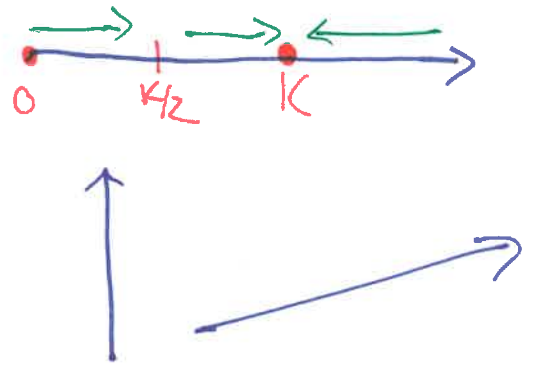
↳ ∘ In ①, $0 < y < K/2$ and so $\frac{dy}{dx}$ is positive: For ex., at $y = K/4$, $\frac{dy}{dx} = r\left(1-\frac{K/4}{K}\right)\left(\frac{K}{4}\right) = r\left(1-1/4\right)\left(\frac{K}{4}\right) = \frac{3rK}{16} > 0$.

∘ In ②, $\frac{K}{2} < y < K$ and $\frac{dy}{dx} > 0$, too: At $y = 3K/4$, for example, $\frac{dy}{dx} = r\left(1-\frac{3K/4}{K}\right)\left(\frac{3K}{4}\right) = r\left(1-3/4\right)\left(\frac{3K}{4}\right) = \frac{3Kr}{16} > 0$.

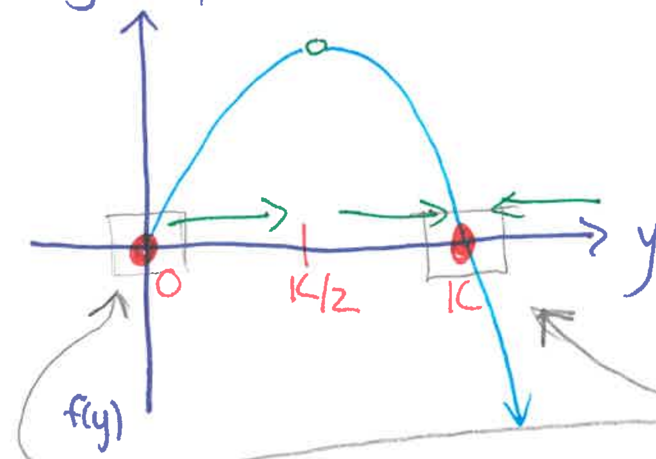
∘ In ③, $y > K$ & $\frac{dy}{dx} < 0$: $y > K \Rightarrow \frac{y}{K} > 1 \Rightarrow 1-\frac{y}{K} < 0$!

(y-axis, cont'd)

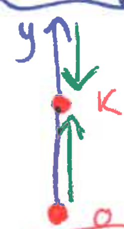
- So, y is increasing for $0 < y < K/2$, & $K/2 < y < K$ and is decreasing for $y > K$. We indicate increasing w/ right arrow and decreasing w/ left arrow:



&



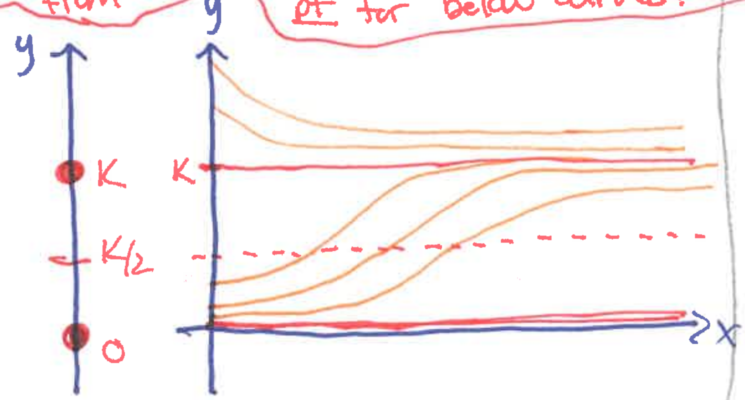
The y-axis is called the phase ~~plane~~ line. We also sometimes draw this vertically:



Note: The vertex $y = K/2$ is where $\frac{dy}{dx}$ goes from incr. to decr.: This is an inflection pt for below curves!

Combining into solutions

- Draw a copy of the phase line to the left of a "normal" xy-plane



- Draw equilibrium solutions, i.e. horiz. lines at $y=0$ & $y=K$.
- Based on green arrows, other solutions are increasing for $0 < y < K$ & decreasing for $y > K$.
- By looking at boxes near $y=0$ and $y=K$, we see that the parabola $f(y)$ is "close to" zero for y "close to" 0 or K. This means that all non-equilibrium solutions will "flatten out" [b/c $f(y) = \frac{dy}{dx}$ near zero] when approaching $y=0$ or $y=K$.
- Draw some curves w/ these features. (orange in above graph)

§ 2.5 : CORRECTION !

- To determine where y is concave $\uparrow$ or concave $\downarrow$, we need to find where $\frac{d^2y}{dx^2} > 0$ & $\frac{d^2y}{dx^2} < 0$, respectively.

- Recall: $\frac{dy}{dx} = f(y)$ where y is a function of x. So:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\big(f(y)\big) \overset{\text{chain}}{\underset{\text{rule}}{=}} f'(y)\cdot\frac{dy}{dx} = f'(y)f(y).$$

$\Rightarrow$ y concave $\uparrow$ when $\frac{d^2y}{dx^2} > 0 \iff f(y)$ & $f'(y)$ have same sign

concave $\downarrow$ when $\frac{d^2y}{dx^2} < 0 \iff f(y)$ & $f'(y)$ have opp. sign.

Ex from last time:

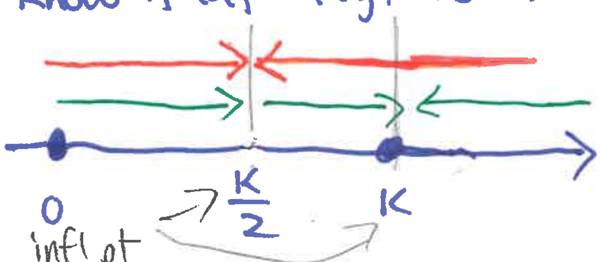
$ry - \frac{ry^2}{K}$

(i) $\frac{dy}{dx} = \underbrace{r\left(1 - \frac{y}{K}\right)y}_{f(y)} \Rightarrow f'(y) = r - \frac{2r}{K}y$

(ii) $f'(y) > 0 \iff r - \frac{2r}{K}y > 0 \iff r > \frac{2r}{K}y \iff y < \frac{K}{2}$

So $f'(y) < 0$ when $y > \frac{K}{2}$.

(iii) Know that $f(y)$ is >0 & <0 as shown on phase line:



= info about f

= info about f'

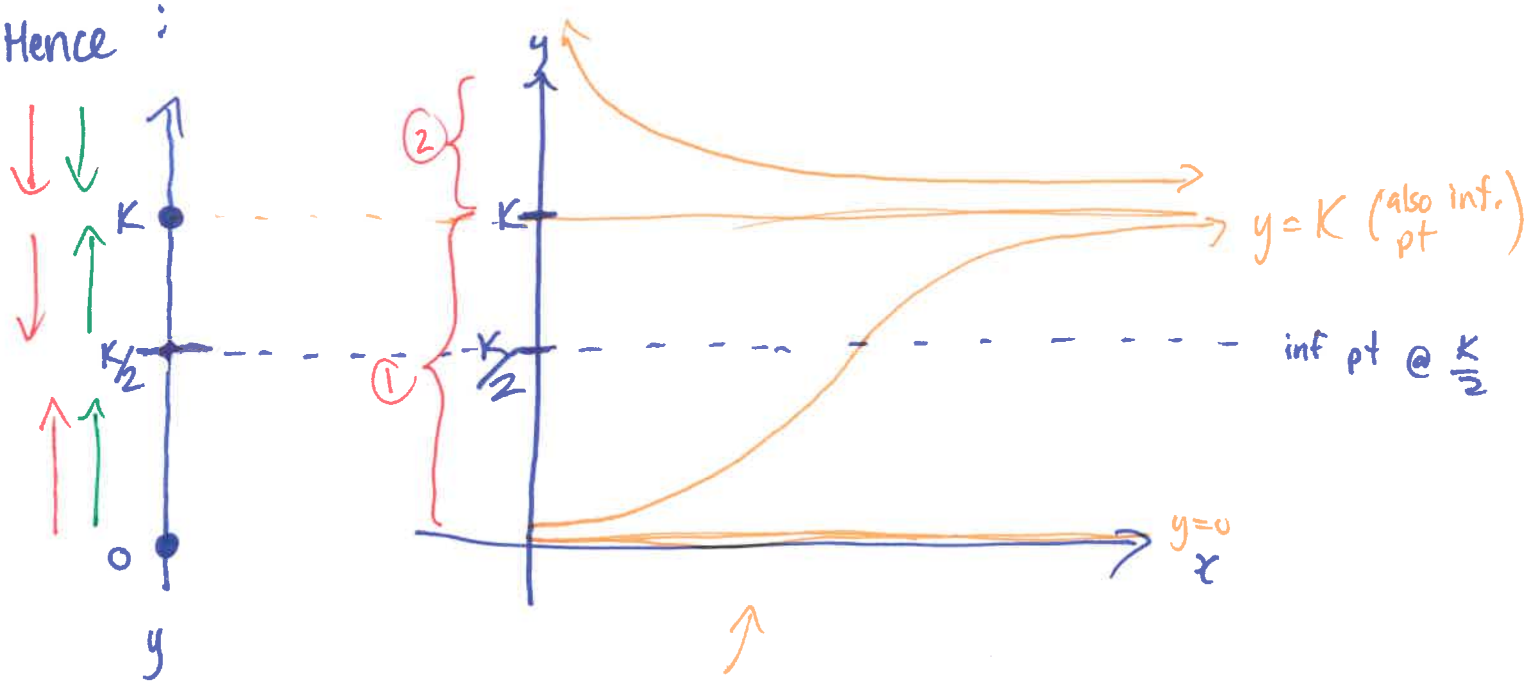
"$\rightarrow$" means ">0"

"$\leftarrow$" means "<0"

(iv) So, y concave up on $(0, K/2) \cup (K, \infty)$ & concave down on $\left(\frac{K}{2}, K\right)$

1(b)

Hence :



These are curves which solve ODE $\frac{dy}{dx} = r(1-\frac{y}{K})y$!

HW: Solve ODE $\frac{dy}{dx} = r(1-\frac{y}{K})y$ using separability + partial fractions !

Note: • In the above pic the "predicted behaviors" occur for initial values in the given intervals

↳ • if IVP $\frac{dy}{dx} = r(1-\frac{y}{K})y$ & $y(x_0)=y_0$ for y_0 in $(0,K)$, then solution has "⌐" shape as in 1

• If IVP ... & $y(x_0)=y_0$ for $y_0 > K$, get "↳" shape like in 2.

• only solution that remains near 0 is $y \equiv 0$

↳ $y=0$ is asymptotically unstable solution.

• Any other y solution satisfies $y \to K$ as $x \to \infty$

↳ $y=K$ is asymptotically stable solution.

4(c)

Notes about these Solutions

- Curves starting above or below the line $y = K/2$ stay there and never intersect it. This is the result of a big theorem we'll study in the next section.

- While we gained this info without solving $\frac{dy}{dx} = r(1-\frac{y}{K})y$, we could have solved it!

$$\frac{dy}{dx} = r\left(1-\frac{y}{K}\right)y \iff \frac{dy}{y(1-\frac{y}{K})} = r\,dx$$

$$\iff \left(\frac{1}{y} + \frac{1/K}{1-y/K}\right)dy = r\,dx$$

using partial fractions:

$$\frac{1}{y(1-y/K)} = \frac{A}{y} + \frac{B}{1-y/K}$$

$\iff 1 = A(1-y/K) + By.$

↳ • @ $y=0$: $1 = A+0$
$\Rightarrow A = 1$

• @ $y=K$: $1 = 0 + BK$
$\Rightarrow B = 1/K$

So, integrating:

$$\ln|y| - \ln\left|1-\frac{y}{K}\right| = rx + C$$

$\Rightarrow \ln\left|\frac{y}{1-y/K}\right| = rx + C$, and if $0<y<K$, we have

$$\ln\left(\frac{y}{1-y/K}\right) = rx + C \iff \frac{y}{1-y/K} = e^{rx+C} = C_2 e^{rx}$$

where $C_2 = e^C$.

Solving:

$$y = C_2 e^{rx}(1-y/K) \iff y + \frac{C_2 e^{rx}}{K}y = C_2 e^{rx} \quad (*)$$

$$\iff \boxed{y = \frac{C_2 e^{rx}}{1+\frac{C_2 e^{rx}}{K}} = \frac{KC_2 e^{rx}}{K+C_2 e^{rx}}}.$$

- You can pick an initial value in the $0<y<K$ range and see that all the predicted things are true.

↳ • Moreover, you can see that as $x\to\infty$, $y\to K$!

Note: w/ some work, $(*)$ can also be shown to be valid for $y>K$.

Def: $y=K$ is an asymptotically stable solution and $y=0$ is a asymptotically unstable solution. (only solution that remains near 0 is $y=0$; all others go toward $y=K$)