§ 2.4 - Differences Between Linear & Nonlinear Equations

For us: "when are solutions to IVP valid?"

separable case: → Do next example instead!

Ex: $\frac{dy}{dx} = \frac{4x - x^3}{4+y^3}$ has solution what?

$y(0) = 1$

↳ $\int 4+y^3\, dy = \int 4x - x^3 dx \Rightarrow 4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$

$\Rightarrow y^4 + 16y - 8x^2 + x^4 = C$

↳ $y(0) = 1 \Rightarrow C = 17$

$\Rightarrow y^4 + 16y + x^4 - 8x^2 = 17 \quad (\star)$

↑ valid everywhere (no holes, discontinuities)

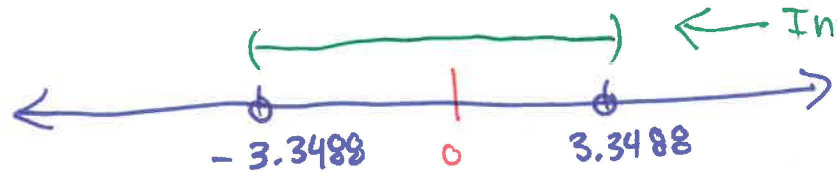
but ...

This solution is only valid when the function is differentiable

↳ From the original ODE, gen. solution not valid when $y^3 + 4 = 0$

$\rightsquigarrow y^3 = -4 \rightsquigarrow y = \sqrt[3]{-4}$. To get an interval, we plug into $(\star)$ & solve for x:

$$x^4 - 8x^2 + \left((-4)^{4/3} + 16(-4)^{1/3} - 17\right) = 0 \Rightarrow x \approx \pm 3.3488$$



← Interval of validity!

↑ initial value!

Ex: $\dfrac{dy}{dx} = \dfrac{4-2x}{4+4y^3}$, $y(0)=1$

$\hookrightarrow (0,1)$

$$\Rightarrow 4y + y^4 = 4x - x^2 + C$$

IVP $\rightsquigarrow$ $5 = C$

$\Longrightarrow$ solution: $x^2 - 4x = 5 - y^4 - 4y$.

only valid (a) $\underbrace{\text{when defined}}_{\text{ALWAYS}}$ & (b) $\underbrace{\text{original ODE defined}}$.

when $4y^3 + 4 \neq 0$

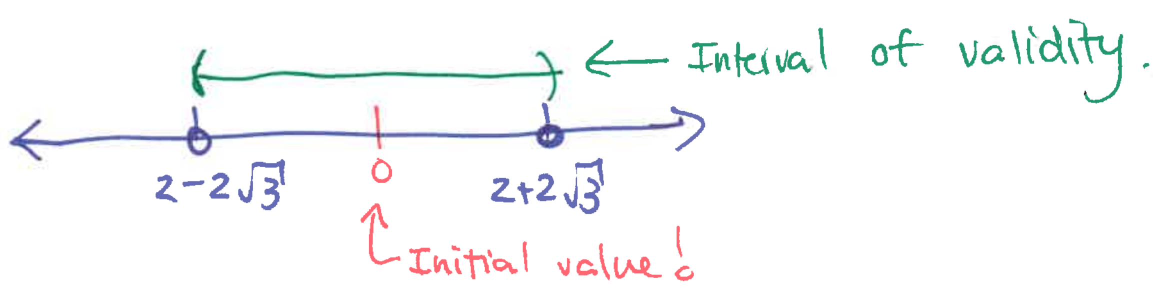
$4y^3 \neq -4$

$y^3 \neq -1 \Rightarrow y \neq -1$.

So: $y=-1$ corresponds to $x^2 - 4x = 5 - 1 + 4$

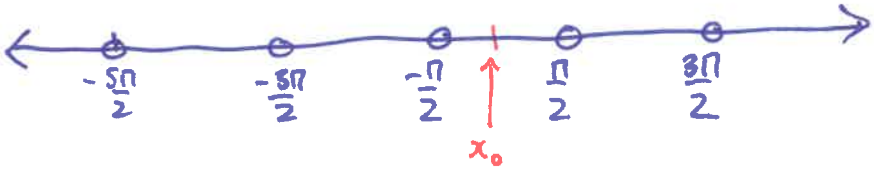
$\Rightarrow x^2 - 4x - 8 = 0$

$\Rightarrow x = 2 \pm 2\sqrt{3}$



Ex: $\frac{dy}{dx} = 3(1+y^2)\sec^2 x$, $y(\underset{x_0}{0}) = 1$

Note: $\frac{dy}{dx}$ undefined when $\cos^2 x = 0 \iff \cos x = 0 \iff x = \frac{\pi}{2}$ & all odd int. multiples of $\pi/2$.



Solve: $\frac{dy}{1+y^2} = 3\sec^2 x \, dx \iff \int \frac{dy}{1+y^2} = \int 3\sec^2 x \, dx$

$\iff \arctan y = 3\tan x + C$

$\iff y = \tan(3\tan x + C)$

$y(0) = 1 \iff 1 = \tan(3\tan(0) + C)$

$\iff 1 = \tan(C)$ $(\star)$

$\Rightarrow C = \frac{\pi}{4}$ $\left(\text{or } \frac{5\pi}{4} \text{ or } \frac{-3\pi}{4} \text{ or} \dots\right)$

So: IVP solution (assuming $C = \frac{\pi}{4}$) is

"inside"

$y = \tan\left(\boxed{3\tan x + \frac{\pi}{4}}\right)$

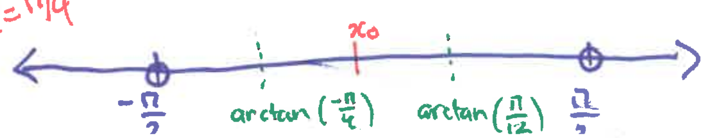
- This is only defined when "inside" lives in one of the above intervals.
- Given x_0, this means: $-\frac{\pi}{2} < \text{inside} < \frac{\pi}{2}$

$\Rightarrow -\frac{\pi}{2} < 3\tan x + \frac{\pi}{4} < \frac{\pi}{2}$

$\Rightarrow -\frac{3\pi}{4} < 3\tan x < \frac{\pi}{4}$

$\Rightarrow -\frac{\pi}{4} < \tan x < \frac{\pi}{12}$

$\Rightarrow \arctan\left(\frac{-\pi}{4}\right) < x < \arctan\left(\frac{\pi}{12}\right)$



What if we pick different C?

- From $(\star)$ $C =$ "$\arctan(1)$", but this has ∞-many vals. (if we look @ unit circle)
- But: $y = \tan(3\tan x + C)$ valid iff $-\frac{\pi}{2} < 3\tan x + C < \frac{\pi}{2}$

$\iff \frac{1}{3}\left(-\frac{\pi}{2} - C\right) < \tan x < \frac{1}{3}\left(\frac{\pi}{2} - C\right)$

$\iff \arctan\left(\frac{1}{3}\left(\frac{-\pi}{2} - C\right)\right) < x < \tan^{-1}\left(\frac{1}{3}\left(\frac{\pi}{2} - C\right)\right)$

can check that our x_0 only works w/ $C = \pi/4$

Linear Case:

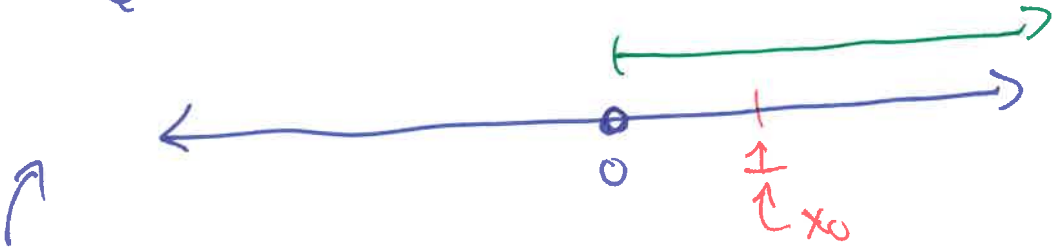
Ex: $xy' + 2y = 4x^2, \quad y(1) = 2$

$\hookrightarrow \; y' + \underbrace{\frac{2}{x}}_{P} y = \underbrace{4x}_{Q}, \quad y(\underbrace{1}_{x_0}) = 2$

Theorem: This IVP has a unique solution on an interval $\alpha < x < \beta$ which contains x_0 and on which both p & q are continuous.

P: Continuous on $(-\infty, 0) \cup (0, \infty)$

Q: continuous on $\mathbb{R}$.



Ans
$\boxed{\text{interval of validity}: (0, \infty)}$

Didn't require you to solve!

Ex: (§2.4 #6)

$(\ln x) y' + y = \cot(x), \quad y(\underbrace{2}_{x_0}) = 3$

$\hookrightarrow \; y' + \underbrace{\frac{1}{\ln x}}_{P} y = \underbrace{\frac{\cot x}{\ln x}}_{Q}$

continuous on their domain

$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos x}{\sin x}$ undefined when $\sin x = 0 \iff x = n\pi$ for n integer

Ans
$\boxed{\text{interval of validity}: (0, 1)}$

Dom(P): $\underbrace{\ln x \neq 0}_{\text{frac}}$ and $\underbrace{x > 0}_{\ln}$ $\Rightarrow$ $\text{dom}(p) = (0,1) \cup (1, \infty)$

Dom(Q): $\underbrace{\ln x \neq 0}_{\text{frac}}$ and $\underbrace{x > 0}_{\ln}$ and $\underbrace{x \neq n\pi}_{\cot}$ $\Rightarrow$ dom(Q) $=$

$(0,1) \cup (1, \pi) \cup (\pi, 2\pi) \cup \dots$