§2.2- Separable ODEs

Recall: To solve an ODE of the form

$$\frac{dy}{dx} = f(x),$$

you simply integrate: $y = \int f(x)\,dx + C$.

- Separable ODEs are a slight generalization of this.

Def: An ODE is separable if it has the form

$$g(y)\,dy = f(x)\,dx$$

for some functions g (w/ only y's) & f (w/ only x's).

Ex: ① $2x = \frac{dy}{dx}$ → separable: $\Rightarrow 1\,dy = 2x\,dx$

② $\frac{3y^2}{2x^4} = \frac{dx}{dy}$ → separable: $\Rightarrow 2x^4 dx = 3y^2 dy$

③ $\frac{dy}{dx} + 2y = x$ → Not separable: No matter what you do, the LHS will have x and/or dx.

Solving separable ODE

To solve the ODE $g(y)\,dy = f(x)\,dx$, you

(a) Integrate

(b) Solve for $y = \ldots$ (if possible)

Ex: ② $\frac{dy}{dx} = \frac{x^2}{1-y^2} \Rightarrow (1-y^2)\,dy = x^2\,dx$ (so separable!)

Now, integrate: $\int (1-y^2)\,dy = \int x^2\,dx$

$$\Rightarrow y - \frac{1}{3}y^3 = \frac{1}{3}x^3 + C. \quad (\star)$$

Here, you can't solve for y, but you can simplify:

flip these.

$$\boxed{(\star) \equiv 3y - y^3 = x^3 + C.}$$

① $dy = x^3(1+y^2)\,dx \Rightarrow \frac{dy}{1+y^2} = x^3\,dx.$ (separable!)

Integrate: $\int \frac{dy}{1+y^2} = \int x^3\,dx$

$$\Rightarrow \tan^{-1}(y) = \frac{1}{4}x^4 + C$$

Solve for y: $\boxed{y = \tan\left(\frac{1}{4}x^4 + C\right).}$

check: y sol'n if derivatives satisfy ODE.

Ans $\Rightarrow \frac{dy}{dx} = \sec^2\left(\frac{1}{4}x^4 + C\right)\cdot(x^3)$

$\Rightarrow \frac{dy}{dx} = x^3\left(1 + \tan^2\left(\frac{1}{4}x^4 + C\right)\right)$

$\Rightarrow \frac{dy}{dx} = x^3(1+y^2)$

$\Rightarrow \frac{dy}{1+y^2} = x^3\,dx.$ ✓

For IVP: $y(0) = 1$

§ 2.2 (Cont'd)

Recall: • The solution to an ODE w/ "C" in it is called a general solution.

• Finding/picking a particular value of "C" yields a particular solution.

Ex: (from last time)

$$(*)\ dy = x^3(1+y^2)\,dx \iff \underbrace{y = \tan\left(\tfrac{1}{4}x^4 + C\right)}_{\text{general solution}}$$

what does this mean geometrically?

↳ simplifying $(*)$ gives $\frac{dy}{dx} = x^3(1+y^2)$. $(**)$

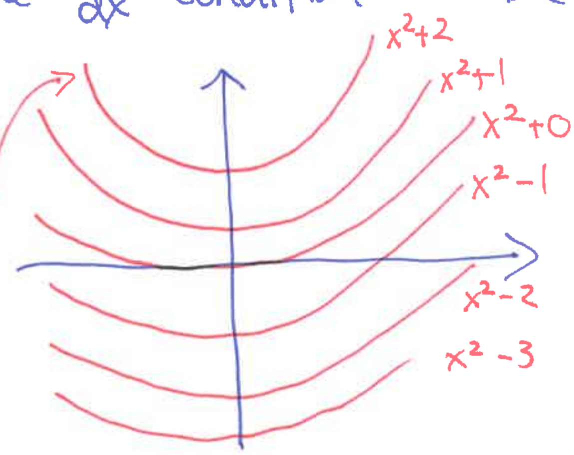
↳ Now, imagine: At every (x,y) pt, you have a small line segment w/ slope $\frac{dy}{dx}$ as given in $(**)$

Ex: @ $(1,1)$: $\frac{dy}{dx} = 1^3(1+1^2) = 1(2) = 2$.

↳ This is called a slope field.

↳ The geometric interpretation of a general solution, then, is an infinite family of curves called integral curves whose tangent line at (x,y) satisfies the $\frac{dy}{dx}$ condition of the ODE.

Ex: $\frac{dy}{dx} = 2x \iff \boxed{y = x^2 + C.}$ general sol



↳ To find the particular solution satisfying $y(0)=2$, we plug in:

$y = x^2 + C \Rightarrow 2 = 0^2 + C$

$\Rightarrow C = 2$

 ⇝ part. sol: $\boxed{y = x^2 + 2.}$