§ 2.1 – Linear ODEs

Ex: Solve $\underbrace{(4+x^2)\frac{dy}{dx} + 2xy}$ $= 4x$

From elementary calculus:

$$\frac{d}{dx}\left[(4+x^2)y\right] = (4+x^2)\frac{dy}{dx} + 2xy$$

So, $\text{LHS} = \frac{d}{dx}\left[(4+x^2)y\right]$

$\Rightarrow$ ODE is: $\frac{d}{dx}\left[(4+x^2)y\right] = 4x$

$$\xrightarrow[\text{WRT } x]{\text{integrate}} \quad (4+x^2)y = 2x^2 + C$$

$$\Rightarrow y = \frac{2x^2 + C}{4+x^2}.$$

So: If one side of an ODE is the derivative (WRT a var. which appears by itself on the other side) of some func., it's easy to solve!

$\hookrightarrow$ Most equations <u>aren't</u>.

Ex: $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{3}e^{t/3}$ $\rightsquigarrow$ LHS Not derivative of anything.

Note: If I multiply everything by $e^{t/2}$:

$$\boxed{e^{t/2}\frac{dy}{dt} + \frac{1}{2}e^{t}y} = \frac{1}{3}e^{t/3}e^{t/2}$$

This equals $\frac{d}{dt}\left[e^{t/2}y\right]$! This can be solved like the first example!

How do we find $e^{t/2}$?

Def: A first order linear ODE is an ODE of the form

$$f(x)\frac{dy}{dx} + g(x)\,y = h(x). \qquad (*)$$

These can always be solved by multiplying by a "magic" integrating factor.

steps: ① Isolate $\frac{dy}{dx}$: $(*)$ becomes

$$\frac{dy}{dx} + \frac{g(x)}{f(x)}\,y = \frac{h(x)}{f(x)} \rightsquigarrow \frac{dy}{dx} + P(x)\,y = Q(x). \qquad (**)$$

② Find the integrating factor $m(x)$:

$$m(x) = e^{\int P(x)\,dx}$$

③ Multiply everything by $m(x)$ & Solve:

$(**)$ becomes: $\frac{dy}{dx}e^{\int P(x)dx} + e^{\int P(x)dx}P(x)\,y = Q(x) \qquad (***)$

$$\Leftrightarrow \underbrace{\frac{d}{dx}\left[e^{\int P(x)\,dx}\,y\right]}\, = Q(x)$$

$$= e^{\int P(x)dx}\frac{dy}{dx} + y \cdot e^{\int P(x)dx} \cdot \frac{d}{dx}\left[\int P(x)\,dx\right]$$

$$= e^{\int P(x)dx}\frac{dy}{dx} + P(x)\,e^{\int P(x)dx}\,y \quad \text{by FTC}$$

$$= \text{LHS } (***)$$

④ Solve: $\frac{d}{dx}\left[e^{\int P(x)\,dx}\,y\right] = Q(x) \Leftrightarrow e^{\int P(x)dx}\,y = \int Q(x)\,dx + C$

$$\Leftrightarrow y = \frac{1}{e^{\int P(x)dx}}\left[\int Q(x)\,dx + C\right]$$

Ex: Solve the IVP

$$xy' + 2y = 4x^2, \quad y(1) = 2.$$

- Find general sol:
  - $xy' + 2y = 4x^2 \iff y' + \boxed{\frac{2}{x}}^{\,p(x)} y = 4x$
  - Now: $m(x) = e^{\int p(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2.$

    ↳ B/c our initial value is @ $(1,2)$, $x \geq 0 \Rightarrow m(x) = x^2$.
  - multiplying ODE by $m(x)$:

$$x^2\left(y' + \frac{2}{x}y\right) = x^2(4x) \iff x^2y' + 2xy = 4x^3$$

$$\iff \frac{d}{dx}\left[x^2 y\right] = 4x^3$$

  - Solve: $x^2 y = x^4 + C$

$$\Rightarrow y = x^2 + \frac{C}{x^2} \quad (*)$$

- Find particular solution:
  - Given $(*)$ & $y(1) = 2$, we get:

$$2 = 1^2 + \frac{C}{1^2} \iff 2 = 1 + C \iff C = 1$$

So: $\boxed{y = x^2 + \frac{1}{x^2}}$ is the solution to the IVP!

Note: This solution isn't valid everywhere!

(we'll revisit in the next section!)

Ex: Solve the IVP $y' - 2y = e^{2x}$, $y(0) = 2$.

- $m(x) = e^{\int -2\,dx} = e^{-2x}$

- Multiply ODE by $m(x)$:

$$e^{-2x}(y' - 2y) = e^{-2x}(e^{2x})$$

$$\Rightarrow e^{-2x}y' - 2e^{-2x}y = 1$$

$$\Rightarrow \frac{d}{dx}\left(e^{-2x}y\right) = 1$$

$$\Rightarrow e^{-2x}y = x + C$$

$$\Rightarrow y = xe^{2x} + Ce^{2x}$$

- Find particular solution: $y(0) = 2 \Rightarrow 2 = 0 + Ce^{0}$

$$\Rightarrow 2 = C$$

$$\rightsquigarrow y = xe^{2x} + 2e^{2x}.$$