Homework 4/test prep 1
(front and back)

Name: ~~KEY~~

(please print neatly!)

Directions: Answer each of the following <u>**three**</u> **(3)** questions, making sure to read the instructions for <u>each question</u> as you proceed. <u>If you use the Laplace table, make sure you specify which entry/entries you're referencing!</u>

Make sure that your submission meets the criteria of the <u>Homework Policy</u> on the `Homework` tab of the course webpage!

Due date: Monday, July 31

1. Find the Laplace transform of each of the following functions.

(a) $f(t) = e^{3t}\sin(4t)$ (Using #9 on table)

$$F(s) = \frac{4}{(s-3)^2+16}$$

(b) $f(t) = e^{3t}\sin(4t) + e^{3t}\cos(4t)$ (#9 + #10)

$$F(s) = \frac{4}{(s-3)^2+16} + \frac{s-3}{(s-3)^2+16}$$

(c) $f(t) = e^{3t}\sin(4t)\cos(4t)$ **Hint**: Use a double-angle formula. $\rightsquigarrow \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$= \frac{1}{2}e^{3t}\sin(8t)$

$$\underset{(\#9)}{\rightsquigarrow} F(s) = \frac{1}{2}\cdot\frac{8}{(s-3)^2+64}$$

(d) $f(t) = \begin{cases} 4.913 & \text{if } 0 \le t < 2 \\ e^{3t}\sin 4t & \text{if } 2 \le t < 5 \\ 2-t & \text{if } 5 \le t < 14 \\ t^2 & \text{if } t \ge 14 \end{cases}$ $F(s) = \int_0^2 e^{-st}(4.913)\,dt + \int_2^5 e^{-st}(e^{3t}\sin 4t)\,dt + \int_5^{14} e^{-st}(2-t)\,dt + \int_{14}^{\infty} e^{-st}t^2\,dt$

$$\Rightarrow F(s) = 4.913\left(\frac{1-e^{-2s}}{s}\right) + \left.\frac{e^{-(s-3)t}\left(-4\cos(4t)-(s-3)\sin(4t)\right)}{s^2-6s+25}\right]_{t=2}^{t=5} + \left.\frac{e^{-st}(1+s(t-2))}{s^2}\right]_{t=5}^{t=14}$$
$$+ \left.-e^{-st}(2+2st+s^2t^2)/s^3\right]_{t=14}^{t=\infty}$$

(e) $f(t) = te^t\sin t$. **Hint**: Let $g(t) = t\sin t$ and rewrite $f(t)$ as $f(t) = e^{ct}g(t)$ for some constant c.

- If $g(t) = t\sin t$, $G(s) = \frac{2s}{(1+s^2)^2}$

$$\underset{(\#14)}{\Longrightarrow} \mathcal{L}\{e^t g(t)\} = G(s-1) = \frac{2(s-1)}{(1+(s-1)^2)^2}$$

2. Use partial fractions to find the inverse Laplace transform of each of the following functions $F(s)$, i.e. find the function $f(t)$ for which $\mathcal{L}\{f(t)\} = F(s)$.

(a) $F(s) = \frac{1}{s^4}$ Using #3:

$$f(t) = \frac{1}{3!}t^3 = \frac{1}{6}t^3$$

(b) $F(s) = \frac{s}{(s+1)(s-1)} = \frac{s}{s^2-1}$

$$f(t) \underset{(\#8)}{=} \cosh(t)$$

(c) $F(s) = \frac{1}{s^2(s+1)^2(s^2+1)^2} \underset{\text{fractions}}{\overset{\text{partial}}{=}} \frac{1}{s^2} - \frac{2}{s} + \frac{1}{4}\cdot\frac{1}{(1+s)^2} + \frac{1}{1+s} + \frac{1}{2}\frac{s}{(1+s^2)^2} + \frac{s}{1+s^2} - \frac{1}{4}\cdot\frac{1}{1+s^2}$

Hint: The denominator consists of repeated factors; brush up on how to handle those!

$$f(t) = t - 2 + \frac{1}{4}te^{-t} + e^{-t} + \frac{1}{4}t\sin t + \cos t - \frac{1}{4}\sin t.$$

(d) $F(s) = \frac{8s^2 - 4s + 12}{s^2(s+1)(s^2+9)} \underset{\text{frac}}{\overset{\text{partial}}{=}} \frac{4}{3}\cdot\frac{1}{s^2} - \frac{16}{9}\cdot\frac{1}{s} + \frac{12}{5}\cdot\frac{1}{s+1} + \frac{16}{15}\cdot\frac{1}{s^2+9}$

$$f(t) = \frac{4}{3}t - \frac{16}{9} + \frac{12}{5}e^{-t} + \frac{16}{45}\sin(3t)$$

(e) $F(s) = 1$ DO NOT GRADE!

Hint: You don't know this function (err..."function"), but...tell me something about what would need to happen to make this true! Try guessing and checking some stuff, etc. etc. Think like a mathematician!

Ans: $f(t) = \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$. This is the Dirac Delta "function" [not a real function; it's a generalized function/distribution]

(f) $F(s) = \frac{s^2}{(s+1)(s-1)} \overset{\text{long division}}{=} 1 + \frac{1}{s^2-1} = 1 + \frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1}$

Hint: Let $\delta(t)$ denote the function from (e) [which you don't know] and use long division!

$$f(t) = \delta(t) + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

3. Solve each of the following IVPs using Laplace transforms; then, check your answers using characteristic equations and/or undetermined coefficients. **Do not check using variation of parameters!**

(a) $y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 1$

$y = \cos(2x) + \frac{3}{2}\sin(2x).$ | check w/ char. equation

(b) $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 3$

$y = e^{3x} - e^{2x}$ | check w/ char. equation.

(c) $y'' - 5y' + 6y = \frac{t^3}{3!}$, $y(0) = 1$, $y'(0) = 1$ **Hint**: Recall: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$y = \frac{1}{1296}\left(\underbrace{2511e^{2x} - 1280e^{3x}}_{\text{char. eq. part}} + 36x^3 + 90x^2 + 114x + 65\right)$

check w/ char. eq & $Y = Ax^3 + Bx^2 + Cx + D$

(d) $y'' + 4y' + 4y = e^t + t$, $y(0) = -2$, $y'(0) = -2$

$y = \frac{1}{36}\left(\underbrace{-67e^{-2x} + -219xe^{-2x}}_{\text{char. eq. part}} + 9x - 9 + 4e^{x}\right)$

check w/ char eq & $Y_1 = Ax + B$, $Y_2 = Ce^x$

(e) $y'' - 8y' - 9y = \cos t - \sin(2t)$, $y(0) = 0$, $y'(0) = -1$

$y = \underbrace{\frac{1}{10}e^{-x} - \frac{1}{10}e^{9x}}_{\text{char eq part.}} - \frac{5}{82}\cos x - \frac{2}{41}\sin x - \frac{16}{425}\cos(2x) + \frac{13}{425}\sin(2x)$

Check w/ char eq & $Y_1 = A\cos x + B\sin x$, $Y_2 = C\cos(2x) + D\sin(2x)$

(f) $y'' + 4y = \cos t - \sin(2t)$, $y(0) = 2$, $y'(0) = -1$

$y = \underbrace{2\cos(2x) - \frac{1}{2}\sin(2x)}_{\text{char. eq. part}} \underbrace{- \frac{1}{3}\cos(2x) - \frac{1}{8}\sin(2x) + \frac{1}{4}x\cos(2x) + \frac{1}{3}\cos(x)}_{\text{May be simplified different}}$

Check w/ char. eq & $Y_1 = A\cos x + B\sin x$, $Y_2 = Cx\cos(2x) + Dx\sin(2x)$

(g) $y'' - 4y = te^{-3t}\cos t - \sin(2t)$, $y(0) = 4$, $y'(0) = 0$ **Hint**: See 1(e) for how to handle $te^{-3t}\cos t$.

$y = \underbrace{2e^{-2x} + 2e^{2x}}_{\text{char. eq. part.}} + \frac{3}{338}e^{2x} \underbrace{- \frac{3}{338}e^{-3x}\cos x + \frac{1}{13}xe^{-3x}\cos x - \frac{41}{338}e^{-3x}\sin x - \frac{3}{26}xe^{-3x}\sin x}_{\text{ditto (f)}}$

Check w/ char eq &: $Y_1 = (Ax+B)e^{-3x}\cos x + (Cx+D)e^{-3x}\sin x$, $Y_2 = E\cos(2x) + F\sin(2x)$