Homework 3/test prep 1

(front and back)

Name: ~ KEY ~

(please print neatly!)

Directions: Answer each of the following four (4) questions, making sure to read the instructions for each question as you proceed.

Make sure that your submission meets the criteria of the Homework Policy on the `Homework` tab of the course webpage!

Note: Questions 1–3 are good quiz prep; all are good exam prep!

Due date: Monday, July 17

1. Solve the initial value problem

$$y'' + 4y = x^2e^{-x} - x\sin x + 4x, \quad y(0) = 0 \quad y'(0) = 1.$$

SOLUTION: Hom: $y''+4y=0 \Longleftrightarrow r^2+4=0 \Longleftrightarrow r=\pm 2i \longleftrightarrow \boxed{c_1\cos(2x)+c_2\sin(2x)}$

Nonhom: • For x^2e^{-x}, guess $(Ax^2+Bx+C)e^{-x} \Rightarrow \ldots \Rightarrow \boxed{A=\frac{1}{5},\ B=4/25,\ C=-2/125}$

• For $-x\sin x$, guess $(Ax+B)\cos x+(Cx+D)\sin x \Rightarrow \ldots \Rightarrow \boxed{A=0,\ B=2/9,\ C=-1/3,\ D=0}$

• For $4x$, guess $Ax+B \Rightarrow \ldots \Rightarrow \boxed{A=1\ \&\ B=0}$

Gen Soln: $y=c_1\cos(2x)+c_2\sin(2x)+\left(\frac{1}{5}x^2+\frac{4}{25}x-\frac{2}{125}\right)e^{-x}+\frac{2}{9}\cos x-\frac{1}{3}x\sin x+x$

IVP: • $c_1-\frac{2}{125}+\frac{2}{9}=0 \Rightarrow \boxed{c_1=\frac{-232}{1125}}$

• $y'=\frac{464}{1125}\sin(2x)+2c_2\cos(2x)-\left(\frac{1}{5}x^2+\frac{4}{25}x-\frac{2}{125}\right)e^{-x}+\left(\frac{2}{5}x+\frac{4}{25}\right)e^{-x}-\frac{2}{9}\sin x$
$-\frac{1}{3}x\cos x-\frac{1}{3}\sin x+1$

$\Rightarrow 1=2c_2+\frac{2}{125}+\frac{4}{25}+1 \Rightarrow \boxed{c_2=\frac{-11}{125}}$

2. Write down the general solution for each of the following non-homogeneous ODEs.

Hint: Do not use undetermined coefficients!

(a) $y'' + 4y' - 5 = 16e^{x/2}$ $\quad r^2+4r-5=0 \Leftrightarrow r=1,\ r=-5 \Leftrightarrow y_1 = e^t,\ y_2 = e^{-5t}$

$W(y_1, y_2) = -6e^{-4t}$

$$y = -e^t \int \frac{(e^{-5t})(16e^{t/2})}{-6e^{-4t}}\,dt + e^{-5t}\int \frac{e^t(16e^{t/2})}{-6e^{-4t}}\,dt$$

$$= \frac{16e^{-t}}{6}\int e^{-t/2}\,dt - \frac{16}{6}e^{-5t}\int e^{11t/2}\,dt \quad \boxed{= \frac{-16}{6}e^{-t}e^{-t/2} - \frac{16}{6}\cdot\frac{2}{11}e^{11t/2}}$$

(b) $2y'' + 8y' + 8y = 2t^{-2}e^{-2t}, \quad t > 0$ $\quad r^2+4r+4=0 \Leftrightarrow r=-2,\ r=-2 \Leftrightarrow y_1 = e^{-2t},\ y_2 = te^{-2t}$

$W(y_1, y_2) = e^{-4t}$

$$y = -e^{-2t}\int \frac{(te^{-2t})(2t^{-2}e^{-2t})}{e^{-4t}}\,dt + te^{-2t}\int \frac{(e^{-2t})(2t^{-2}e^{-2t})}{e^{-4t}}\,dt$$

$$= -2e^{-2t}\int t^{-1}\,dt + 2te^{-2t}\int t^{-2}\,dt \quad \boxed{= -2e^{-2t}\log(t) - 2e^{-2t}}$$

(c) $y'' - 2y' + y = 3\sec(2t), \quad t < \frac{\pi}{6}$ $\quad r=1,\ r=1 \Rightarrow y_1 = e^t,\ y_2 = te^t$

$W(y_1, y_2) = e^{2t}$

$$y = -e^t\int \frac{te^t(3\sec 2t)}{e^{2t}}\,dt + te^t\int \frac{e^t(3\sec 2t)}{e^{2t}}\,dt$$

$$\boxed{= -3e^t\int te^{-t}\sec(2t)\,dt + 3te^t\int e^{-t}\sec(2t)\,dt}$$

(d) $y'' - 5y' + 6y = g(t)$ **Hint**: $g(t)$ is an arbitrary continuous function.

$r^2-5r+6=0 \rightarrow (r-3)(r-2)=0 \rightarrow r=3,\ r=2$ $\quad y_1 = e^{3t} \quad y_2 = e^{2t}$

$W(y_1, y_2) = -e^{5t}$

$$y = -e^{3t}\int \frac{e^{2t}g(t)}{-e^{5t}}\,dt + e^{2t}\int \frac{e^{3t}g(t)}{-e^{5t}}\,dt$$

$$\boxed{= e^{3t}\int e^{-3t}g(t)\,dt - e^{2t}\int e^{-2t}g(t)\,dt}$$

3. Show that the functions y_1 and y_2 satisfy the corresponding homogeneous equation; then, find a particular solution of the given non-homogeneous ODE. Throughout, assume $x > 0$.

$$x^2y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\sin(x); \quad y_1 = \frac{\sin x}{\sqrt{x}}, \quad y_2 = \frac{\cos x}{\sqrt{x}}$$

SOLUTION: To show part (i)

$$y_1 = \frac{\sin(x)}{\sqrt{x}} \Rightarrow y_1' = \frac{\cos(x)}{\sqrt{x}} - \frac{\sin(x)}{2x^{3/2}}$$

$$y_1'' = -\frac{\cos(x)}{x^{3/2}} + \frac{3\sin(x)}{4x^{5/2}} - \frac{\sin(x)}{\sqrt{x}}$$

$$\Rightarrow x^2y_1'' + xy_1' + (x^2 - 0.25)y_1 = 0.$$

$$y_2 = \frac{\cos x}{\sqrt{x}} \Rightarrow y_2' = -\frac{\cos(x)}{2x^{3/2}} - \frac{\sin(x)}{\sqrt{x}}$$

$$\Rightarrow y_2'' = \frac{3\cos(x)}{4x^{5/2}} - \frac{\cos(x)}{\sqrt{x}} + \frac{\sin(x)}{x^{3/2}}$$

$$\Rightarrow x^2y_2'' + xy_2' + (x^2 - 0.25)y_2 = 0$$

$$W(y_1, y_2) = \frac{-1}{x}$$

$$\Rightarrow Y(x) = \frac{-\sin(x)}{\sqrt{x}} \int \frac{\frac{\cos(x)}{\sqrt{x}} \cdot 3x^{3/2}\sin(x)}{-1/x}\,dx + \frac{\cos(x)}{\sqrt{x}} \int \frac{\frac{\sin(x)}{\sqrt{x}} \cdot 3x^{3/2}\sin(x)}{-1/x}\,dx$$

$$= \frac{-\sin(x)}{\sqrt{x}} \int \left(-3x^2\sin(x)\cos(x)\right)dx + \frac{\cos(x)}{\sqrt{x}} \int \left(-3x^2\sin^2(x)\right)dx$$

$$= \frac{-\sin(x)}{\sqrt{x}} \int \frac{-3}{2}x^2\sin(2x)\,dx + \frac{\cos(x)}{\sqrt{x}} \int -3x^2\sin^2x\,dx$$

$\vdots$ IBP twice $\qquad\qquad$ $\vdots$ IBP twice

$$= \frac{-\sin(x)}{\sqrt{x}} \left(\frac{-3}{8}\cos(2x) - \frac{3}{4}x\sin(2x) + \frac{3}{4}x^2\cos(2x)\right)$$

$$+ \frac{\cos(x)}{\sqrt{x}} \left(-\frac{x^3}{2} - \frac{3}{8}\sin(2x) + \frac{3}{4}x\cos(2x) + \frac{3}{4}x^2\sin(2x)\right)$$

4. Find the Laplace transform for each of the following functions. Throughout, assume that a and b are real constants and that $i = \sqrt{-1}$ is the imaginary unit.

(a) $f(t) = 1$

$$\int_0^\infty e^{-st}(1)\,dt = \frac{-1}{s}e^{-st}\Big]_0^\infty = \boxed{\frac{1}{s}}$$

(b) $f(t) = t^2$

$$\int_0^\infty e^{-st}(t^2)\,dt$$

IBP twice $= \ldots = -\dfrac{e^{-st}(2+2st+s^2t^2)}{s^3}\Big]_0^\infty = \dfrac{-0 - -1(2)}{s^3} = \boxed{\dfrac{2}{s^3}}$

(c) $f(t) = \sin(bt)$ **Hint**: $\sin(bt) = \dfrac{e^{ibt} - e^{-ibt}}{2i}$

$\displaystyle\int_0^\infty e^{-st}\sin(bt)\,dt = \int_0^\infty e^{-st}\left(\frac{e^{ibt}-e^{-ibt}}{2i}\right)dt = \cdots$ see lecture notes $= \boxed{\dfrac{b}{s^2+b^2}}$

(d) $f(t) = t^2e^{at}$ **Hint**: Use integration by parts!

$\displaystyle\int_0^\infty e^{-st}t^2e^{at}\,dt = \int_0^\infty t^2e^{-t(s-a)}\,dt =$ IBP twice $\ldots = \dfrac{e^{-t(s-a)}\left(2+(a-s)t(-2+at-st)\right)}{(a-s)^3}\Big]_0^\infty$

$$= \boxed{\frac{-2}{(a-s)^3}}$$

(e) $f(t) = 5\sin(bt) - 2t^2e^{at}$

$$\int_0^\infty e^{-st}\left(5\sin(bt) - 2t^2e^{at}\right)dt = 5\int_0^\infty e^{-st}\sin(bt)\,dt - 2\int_0^\infty t^2e^{at}\,dt$$

$$\boxed{= 5\left(\frac{b}{s^2+b^2}\right) - 2\left(\frac{-2}{(a-s)^3}\right).}$$