Homework 2
(front and back)

Name: ~ KEY ~

(please print neatly!)

Directions: Answer each of the following six (6) questions, making sure to read the instructions for each question as you proceed.

Make sure that your submission meets the criteria of the Homework Policy on the Homework tab of the course webpage!

Due date: Friday, July 7

p: $(-\infty,0)\cup(0,3/2)\cup(3/2,2)\cup(2,\infty)$
q: $(1,3/2)\cup(3/2,2)\cup(2,e)\cup(e,\infty)$
g: $(0,1/2)\cup(1/2,3/2)\cup(3/2,2)\cup(2,3)\cup(3,\infty)$

Together: $(1,3/2)\cup(3/2,2)\cup(2,e)\cup(e,3)\cup(3,\infty)$

1. Consider the second-order linear IVP

$$(2x^2-7x+6)y''+\left(\frac{1}{2^x-1}\right)y'-\left(\frac{3\ln x}{\ln(\ln x)}\right)y=\frac{9\sqrt{x+1}}{(\ln 2x)\,(\ln x/3)},\quad y(x_0)=y_0,\quad y'(x_0)=y_0'.$$

For each of the following values x_0, state the largest interval on which the corresponding IVP has a unique solution or state that no solution exists.

(a) $x_0=2$

DNE

(b) $x_0=\dfrac{3+e}{2}$

$(e,3)$

(c) $x_0=\dfrac{-2.1}{4}$

DNE

(d) $x_0=e$

DNE

(e) $x_0=\dfrac{1}{2}$

DNE

(f) $x_0=\pi$

$(3,\infty)$

(g) $x_0=-1$

DNE

(h) $x_0=\dfrac{3}{2}$

DNE

(i) $x_0=\dfrac{3.1}{2}$

$(\frac{3}{2},2)$

(j) $x_0=0$

DNE

(k) $x_0=\dfrac{9e}{10}$

$(2,e)$

(l) $x_0=5$

$(3,\infty)$

2. Find the Wronskian of the two solutions y_1 and y_2 of each of the following second-order linear ODEs. Do not attempt to solve the ODEs! use Abel!

(a) $y'' - \frac{2}{x}y' + 3xy = 0$

$W = C\exp\left(-\int -\frac{2}{x}\,dx\right) = C\exp(2\ln|x|) = C|x|^2 = Cx^2$

(b) $e^x y'' - (e^{2x}\sin x)\, y' - e^x y = 0 \Rightarrow p(x) = -e^x \sin x$

$W = C\exp\left(-\int -e^x \sin x\, dx\right) = C\exp\left(\int e^x \sin x\, dx\right) = C\exp\left(\frac{e^x \sin x - e^x \cos x}{2}\right)$

(c) $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ where $\alpha = \text{const}$ $\rightarrow p(x) = \frac{1}{x}$

$W = C\exp\left(-\int \frac{1}{x}\, dx\right) = C\exp(-\ln|x|) = \frac{C}{|x|} = \frac{C}{x}$

3. A second-order linear homogeneous ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be *(second-order) exact* if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0 \tag{1}$$

for some function $f(x)$. A well-known result in the theory of second-order ODEs is that $P(x)y'' + Q(x)y' + R(x)y = 0$ is (second-order) exact if and only if $P''(x) - Q'(x) + R(x) = 0$.

(a) Show that $xy'' - (\cos x)y' + (\sin x)y = 0$ is (second-order) exact.

$P'' - Q' + R = (x)'' - (-\cos x)' + \sin x = 0 - \sin x + \sin x = 0$!

(b) Rewrite $xy'' - (\cos x)y' + (\sin x)y = 0$ in the form (1). **You don't know what $f(x)$ is yet!**

$[xy']' + [f(x)y]' = 0$ $\left(\Rightarrow xy'' + y' + f(x)y' + f'(x)y = 0\right)$

(c) Find $f(x)$ by expanding the left-hand side (LHS) of part (b) and comparing it term-by-term with the ODE $xy'' - (\cos x)y' + (\sin x)y = 0$. We have $xy'' + y'(1+f(x)) + f'(x)y = xy'' - \cos x\, y' + \sin x\, y$

$\Rightarrow 1 + f(x) = -\cos x \rightarrow \boxed{f(x) = -\cos x - 1}$

(d) Reduce $xy'' - (\cos x)y' + (\sin x)y = 0$ to a first-order linear ODE by integrating both sides of the result from (b) with respect to x. **Don't forget to plug in $f(x)$ from (c)!**

$\int 0\,dx = \int \left([xy']' + [(-\cos x - 1)y]'\right)dx \Rightarrow \boxed{xy' + (-\cos x - 1)y = \text{const}.}$ $\left(\Rightarrow y' + \frac{-\cos x - 1}{x}y = \text{const}\right)$

(e) The result from part (d) is a first-order linear ODE. Use it to solve for y by

- finding and multiplying both sides by its integrating factor (see §2.1 for a refresher); and
- finding its general solution (ditto §2.1). $m(x) = \exp\left(\int \frac{-\cos x - 1}{x}\, dx\right)$ [can't simplify]

$\Rightarrow m(x)y' + m(x)\frac{-\cos x - 1}{x}y = \text{const}\cdot m(x) \Rightarrow \frac{d}{dx}[m(x)y] = \text{const}\cdot m(x).$

$\Rightarrow m(x)y = \int \text{const}\cdot m(x)\,dx \Rightarrow \boxed{y = \frac{1}{m(x)}\int c\cdot m(x)\,dx.}$

4. For each of the following non-homogeneous ODEs, find the undetermined coefficients A, B, C, ... which make the indicated "guess" function $Y(x)$ a particular solution.

(a) $y'' - 2y' - 2y = 2x + 4x^3$;
Guess: $Y(x) = Ax^3 + Bx^2 + Cx + D$

$A = -2 \quad B = 6 \quad C = -19 \quad D = 25$

(b) $y'' + 2y' - 3y = 4\sin 2x$;
Guess: $Y(x) = A\cos 2x + B\sin 2x$

$A = \frac{-16}{65} \quad B = \frac{-28}{65}$

(c) $y'' + 9y = 6$;
Guess: $Y(x) = A$

$A = 2/3$

(d) $y'' + 9y = x^2 e^{3x}$;
Guess: $Y(x) = (Ax^2 + Bx + C)e^{3x}$

$A = \frac{1}{18} \quad B = \frac{-1}{27} \quad C = \frac{1}{162}$

$0 = y'' + 9y \iff r^2 + 9 = 0 \iff r = \pm 3i$

5. Using parts (c) and (d) above, find the general solution of the ODE $y'' + 9y = x^2e^{3x} + 6$.

Hint: If the right-hand side (RHS) of a non-homogeneous ODE has the form $f(x) + g(x)$, you can

- split up the RHS;
- use two guesses—one called $Y_1(x)$ (corresponding to RHS=$f(x)$) and the other $Y_2(x)$ (for RHS=$g(x)$);
- find the undetermined coefficients for each Y_1 and Y_2; and
- form the sum $Y_3 = Y_1 + Y_2$.

Y_3 (with the "undetermined coefficients" determined and plugged-in) will be the solution you seek!

$Y_3 = \left(\frac{1}{18}x^2 - \frac{1}{27}x + \frac{1}{162}\right)e^{3x} + 2/3$

$\boxed{\text{gen soln: } Y = C_1\cos(3x) + C_2\sin(3x) + Y_3}$, where $Y_3 =$

6. Sometimes, the method of undetermined coefficients will lead you to guess a particular solution $Y(x)$ to the non-homogeneous ODE $ay'' + by' + cy = g(x)$ which is *also* a solution to the corresponding homogeneous ODE $ay'' + by' + cy = 0$. When this is true, we need a way to come up with a new guess.

   (a) Show that $y_1 = e^{-x}$ and $y_2 = e^{4x}$ form a fundamental system of solutions for the homogeneous ODE $y'' - 3y' - 4y = 0$.

   y_1 & y_2 solve the ODE & $W(y_1, y_2) \neq 0$.

   [you show this!]

   (b) Write down a reasonable guess $Y(x)$ (with undetermined coefficients) for a particular solution of the non-homogeneous ODE $y'' - 3y' - 4y = 2e^{-x}$. **Hint**: The answer is $Y(x) = Ae^{-x}$; now explain why.

   $Y = Ae^{-x}$ b/c RHS has e^{-x}.

   (c) Using your Y from part (b), show that no combination of A, B, C, ... yields a valid particular solution having the form you guessed.

   Show it, or note that Ae^{-x} already included in y_1.

When you run into a situation like the above, a good new guess is x times the thing you guessed before.

   (d) Using the guess $Y(x) = Axe^{-x}$, find the coefficient A which makes Y a particular solution of the ODE $y'' - 3y' - 4y = 2e^{-x}$.

   $A = \frac{-2}{5}$

Sometimes, both your first guess *and* x times your first guess will be elements of a fundamental system of solutions for the corresponding homogeneous ODE. This means you'll need *another* new guess, and the next obvious choice is x^2 times the thing you guessed first.

   (e) Show that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ form a fundamental system of solutions for the homogeneous ODE $y'' + 2y' + y = 0$.

   Same as (a) [you do the work, though!]

   (f) Write down **two** reasonable guesses $Y_1(x)$ and $Y_2(x)$ (both with undetermined coefficients) for a particular solution of the non-homogeneous ODE $y'' + 2y' + y = 2e^{-x}$. **Hint**: The answers are $Y_1(x) = Ae^{-x}$ and $Y_2(x) = Axe^{-x}$; once again, explain why.

   RHS $= 2e^{-x} \Rightarrow$ guess 1 $= Ae^{-x}$; this covered by y_1, so guess 2 $= Axe^{-x}$.

   (g) Using the guess $Y(x) = Ax^2e^{-x}$, find the coefficient A which makes Y a particular solution of the ODE $y'' + 2y' + y = 2e^{-x}$.

   $A = 1$