Homework 2

(front and back)

Name: ______________________________

(please print neatly!)

Directions: Answer each of the following **six (6)** questions, making sure to read the instructions for each question as you proceed.

Make sure that your submission meets the criteria of the Homework Policy on the `Homework` tab of the course webpage!

Due date: Friday, July 7

1. Consider the second-order linear IVP

$$(2x^2 - 7x + 6)y'' + \left(\frac{1}{2^x - 1}\right)y' - \left(\frac{3\ln x}{\ln(\ln x)}\right)y = \frac{9\sqrt{x+1}}{(\ln 2x)\,(\ln x/3)}, \quad y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

For each of the following values x_0, state the largest interval on which the corresponding IVP has a unique solution or state that no solution exists.

(a) $x_0 = 2$

(b) $x_0 = \dfrac{3+e}{2}$

(c) $x_0 = \dfrac{-2.1}{4}$

(d) $x_0 = e$

(e) $x_0 = \dfrac{1}{2}$

(f) $x_0 = \pi$

(g) $x_0 = -1$

(h) $x_0 = \dfrac{3}{2}$

(i) $x_0 = \dfrac{3.1}{2}$

(j) $x_0 = 0$

(k) $x_0 = \dfrac{9e}{10}$

(l) $x_0 = 5$

2. Find the Wronskian of the two solutions y_1 and y_2 of each of the following second-order linear ODEs. Do not attempt to solve the ODEs!

   (a) $y'' - \dfrac{2}{x}y' + 3xy = 0$

   (b) $e^x y'' - \left(e^{2x}\sin x\right)y' - e^x y = 0$

   (c) $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ where $\alpha = \text{const}$

3. A second-order linear homogeneous ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ is said to be *(second-order) exact* if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0 \tag{1}$$

   for some function $f(x)$. A well-known result in the theory of second-order ODEs is that $P(x)y'' + Q(x)y' + R(x)y = 0$ is (second-order) exact if and only if $P''(x) - Q'(x) + R(x) = 0$.

   (a) Show that $xy'' - (\cos x)y' + (\sin x)y = 0$ is (second-order) exact.

   (b) Rewrite $xy'' - (\cos x)y' + (\sin x)y = 0$ in the form (1). **You don't know what $f(x)$ is yet!**

   (c) Find $f(x)$ by expanding the left-hand side (LHS) of part (b) and comparing it term-by-term with the ODE $xy'' - (\cos x)y' + (\sin x)y = 0$.

   (d) Reduce $xy'' - (\cos x)y' + (\sin x)y = 0$ to a first-order linear ODE by integrating both sides of the result from (b) with respect to x. **Don't forget to plug in $f(x)$ from (c)!**

   (e) The result from part (d) is a first-order linear ODE. Use it to solve for y by
      - finding and multiplying both sides by its integrating factor (see §2.1 for a refresher); and
      - finding its general solution (ditto §2.1).

4. For each of the following non-homogeneous ODEs, find the undetermined coefficients A, B, C, $\ldots$ which make the indicated "guess" function $Y(x)$ a particular solution.

   (a) $y'' - 2y' - 2y = 2x + 4x^3$;
   Guess: $Y(x) = Ax^3 + Bx^2 + Cx + D$

   (b) $y'' + 2y' - 3y = 4\sin 2x$;
   Guess: $Y(x) = A\cos 2x + B\sin 2x$

   (c) $y'' + 9y = 6$;
   Guess: $Y(x) = A$

   (d) $y'' + 9y = x^2 e^{3x}$;
   Guess: $Y(x) = (Ax^2 + Bx + C)e^{3x}$

5. Using parts (c) and (d) above, find the general solution of the ODE $y'' + 9y = x^2 e^{3x} + 6$.

   Hint: If the right-hand side (RHS) of a non-homogeneous ODE has the form $f(x) + g(x)$, you can
   - split up the RHS;
   - use two guesses—one called $Y_1(x)$ (corresponding to RHS=$f(x)$) and the other $Y_2(x)$ (for RHS=$g(x)$);
   - find the undetermined coefficients for each Y_1 and Y_2; and
   - form the sum $Y_3 = Y_1 + Y_2$.

   Y_3 (with the "undetermined coefficients" determined and plugged-in) will be the solution you seek!

6. Sometimes, the method of undetermined coefficients will lead you to guess a particular solution $Y(x)$ to the non-homogeneous ODE $ay'' + by' + cy = g(x)$ which is *also* a solution to the corresponding homogeneous ODE $ay'' + by' + cy = 0$. When this is true, we need a way to come up with a new guess.

   (a) Show that $y_1 = e^{-x}$ and $y_2 = e^{4x}$ form a fundamental system of solutions for the homogeneous ODE $y'' - 3y' - 4y = 0$.

   (b) Write down a reasonable guess $Y(x)$ (with undetermined coefficients) for a particular solution of the non-homogeneous ODE $y'' - 3y' - 4y = 2e^{-x}$. **Hint**: The answer is $Y(x) = Ae^{-x}$; now explain why.

   (c) Using your Y from part (b), show that no combination of A, B, C, ... yields a valid particular solution having the form you guessed.

When you run into a situation like the above, a good new guess is x times the thing you guessed before.

   (d) Using the guess $Y(x) = Axe^{-x}$, find the coefficient A which makes Y a particular solution of the ODE $y'' - 3y' - 4y = 2e^{-x}$.

Sometimes, both your first guess *and* x times your first guess will be elements of a fundamental system of solutions for the corresponding homogeneous ODE. This means you'll need *another* new guess, and the next obvious choice is x^2 times the thing you guessed first.

   (e) Show that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ form a fundamental system of solutions for the homogeneous ODE $y'' + 2y' + y = 0$.

   (f) Write down **two** reasonable guesses $Y_1(x)$ and $Y_2(x)$ (both with undetermined coefficients) for a particular solution of the non-homogeneous ODE $y'' + 2y' + y = 2e^{-x}$. **Hint**: The answers are $Y_1(x) = Ae^{-x}$ and $Y_2(x) = Axe^{-x}$; once again, explain why.

   (g) Using the guess $Y(x) = Ax^2e^{-x}$, find the coefficient A which makes Y a particular solution of the ODE $y'' + 2y' + y = 2e^{-x}$.