Tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Throughout, let f be a function satisfying $f(n) = a_n$.

The first test you should ever do, always

The "is convergence even possible?!" test:

- *What you know:*
  - If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, your series **cannot** converge (i.e., it must diverge).
  - If $\lim_{n\to\infty} a_n = 0$, your series **may** converge.

Specialty "tests" that work only for very specific cases

1. Geometric series:
   - Can only use when your series looks like $\sum_{n=0}^{\infty} ar^n$ and/or $\sum_{n=1}^{\infty} ar^{n-1}$ for constants $a \neq 0$ and r.
   - *What you know:*
     - Your series converges **if and only if** $|r| < 1$.
     - If your series converges, it converges to $\frac{a}{1-r}$
2. p-series:
   - Can only use when your series looks like $\sum \frac{1}{n^p}$ for p a constant.
   - *What you know:*
     - Your series converges **if and only if** $p > 1$.

Tests that work for series with ***positive terms only***

1. The integral test:
   - Can only use when f is positive, decreasing, and continuous, and when $\int_1^{\infty} f(x)\,dx$ can be (hopefully-easily) computed.
   - *What you know:*
     - $\sum_{n=1}^{\infty} a_n$ converges **if and only if** $\int_1^{\infty} f(x)\,dx$ converges.
2. The comparison test:
   - Can only use when $\{a_n\}$ "looks like" another sequence $\{b_n\}$ where $\{b_n\}$ also has positive terms and where convergence/divergence of $\sum b_n$ is known.
   - **Usually**, you want to pick $\{b_n\}$ so that $\sum b_n$ is either a geometric series or a p-series.
   - *What you know:*
     - If $a_n \leq b_n$ for all n and if $\sum b_n$ converges, then $\sum a_n$ **converges.**
     - If $a_n \geq b_n$ for all n and if $\sum b_n$ diverges, then $\sum a_n$ **diverges.**
     - **Note:** Knowing that $a_n \geq b_n$ for $\sum b_n$ convergent *or* that $a_n \leq b_n$ for $\sum b_n$ divergent tells you **nothing**!

3. The limit comparison test:
   - As with the comparison test, you can use this when $\{a_n\}$ "looks like" another sequence $\{b_n\}$ where $\{b_n\}$ also has positive terms and where convergence/divergence of $\sum b_n$ is known.

     However: This test should be used either (a) when the comparison test **doesn't** work, or (b) if you don't like the comparison test (because inequalities are hard). *If the comparison test works, the limit comparison test will work, but not vice versa!*
   - As with the comparison test, you want to pick $\{b_n\}$ so that $\sum b_n$ is either a geometric series or a p-series.
   - *What you know:*
     - If $\lim_{n\to\infty} a_n/b_n = c$ where $0 < c < \infty$, then either $\sum a_n$ and $\sum b_n$ *both* converge or they *both* diverge.
     - If $\lim_{n\to\infty} a_n/b_n = c$ where $c = 0$ or $c = \infty$, then you know **nothing!** In particular: You may have "squinted wrong" when picking $\{b_n\}$ or your series $\sum a_n$ may genuinely diverge. *You just don't know!*

Tests that work *only* for series with negative terms

The alternating series test:

- To use, $\sum a_n$ must be an *alternating series*, i.e. $\sum a_n$ must be writable as $\sum(-1)^n b_n$ or $\sum(-1)^{n+1} b_n$ for $\{b_n\}$ a sequence with positive terms.
- *What you know:*
  - If b_n is decreasing (i.e. if $b_{n+1} \leq b_n$ for all n) if $\lim_{n\to\infty} b_n = 0$, then $\sum a_n$ **converges**.

Other tests

1. The ratio test:
   - Can use with *any* series.
   - *What you know:*
     - If $\lim_{n\to\infty} |a_{n+1}/a_n| < 1$, then $\sum a_n$ **converges absolutely** (and hence **converges**).
     - If $\lim_{n\to\infty} |a_{n+1}/a_n| > 1$ (or is ∞), then $\sum a_n$ **diverges**.
     - If $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, then $\sum a_n$ the ratio test tells you **nothing!**
2. The root test:
   - Can use with *any* series.
   - *What you know:*
     - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ **converges absolutely** (and hence **converges**).
     - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or is ∞), then $\sum a_n$ **diverges**.
     - If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then $\sum a_n$ the root test tells you **nothing!**