How to determine whether $\sum_{n=1}^{\infty} a_n$ converges or diverges.

Throughout, let f be a function satisfying $f(n) = a_n$.

Question 1: *Can* my series converge (i.e. does $\lim_{n\to\infty} a_n$ exist *and* does $\lim_{n\to\infty} a_n = 0$?)

- If *no*: You're done; $\sum_{n=1}^{\infty} a_n$ diverges.
- If *yes*: Your series *may* converge. **Go to Question 2**.

Question 2: Does my series have negative terms?

- If *no*: You have a positive series. **Go to Question 3**.
- If *yes*: Go to **Question 5**.

Question 3: Is my series a geometric series or a p-series?

- If *yes*: Use the info you know about **geometric series** and/or ***p*-series** and you're done.
- If *no*: Go to **Question 4**.

Question 4: If I squint at my series, does it kinda-sorta look like a geometric series or a p-series?

- If *yes*, use either **the comparison test** or **the limit comparison test**.
  - Use **the comparison test** if you can get the inequalities to work.
  - Use **the limit comparison test** if you *can't* get the inequalities to work **but** you're sure you're squinting is accurate.
- If *no*:
  - Does my series have factorials and/or $(\text{constant})^n$?

    $\Longrightarrow$ **Use the Ratio Test!**
  - Does a_n have the form $a_n = (b_n)^n$ (a whole function to the nth power)?

    $\Longrightarrow$ **Use the Root Test!**
  - Does it look like I can find $\int_1^{\infty} f(x)\,dx$?

    $\Longrightarrow$ **(Try to) Use the Integral Test!** (f must be continuous, positive, and decreasing!)
  - If none of the ratio, root, or integral tests seem appropriate:

    $\Longrightarrow$ Ask whatever higher power you believe in for an intervention. (If you don't have a higher power, ask a friend to borrow theirs.)

Question 5: Is my series alternating? (i.e., is $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $\{b_n\}$ has all positive terms?)

- If *yes*: (Try to) Use the **Alternating Series Test!** (b_n must be decreasing and $\lim_{n\to\infty} b_n = 0$ must hold)
- If *no*:
  - Does my series have factorials and/or $(\text{constant})^n$?

    $\Longrightarrow$ **Use the Ratio Test!**
  - Does a_n have the form $a_n = (b_n)^n$ (a whole function to the nth power)?

    $\Longrightarrow$ **Use the Root Test!**
  - If neither the ratio nor root test seems applicable:

    $\Longrightarrow$ See **Question 4** about borrowing higher powers, etc.

    $\Longrightarrow$ Try looking at $\sum_{n=1}^{\infty} |a_n|$ directly by going back at **Question 3**.

Notes:

† In **Question 4**, it's important to recognize what "squint" means.

For example: $\sum_{n=1}^{\infty} \frac{n}{2n^3+1}$ kinda-sorta looks like a p-series with $p = 2$ while $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ looks a smidgen like a geometric series with $a = 16/3$ and $r = 4/3$. How do we get that? **By picking the highest power of n on the top and bottom and ignoring everything else!**

† Stuff to note about the *yes* in **Question 4**:

- If the comparison test works, the limit comparison test definitely works.
- When both work, here's what to keep in mind:
  - The comparison test = easy algebra but perhaps difficult intuition (*WTF do I compare with?!*...see the comparison test for integrals).
  - The limit comparison test = easy intuition (just keep the highest power on top and the highest power on bottom) but messy algebra (because dividing + limits).
- Sometimes, the limit comparison test works but the comparison test doesn't:

  Example: We **know** that $\sum 1/2^n$ converges (it's a geometric series with $r = 1/2$).

  Moreover, because $2^n < 2^n + 1 \implies 1/(2^n + 1) < 1/2^n$, we can use **the comparison test** to conclude that $\sum 1/(2^n + 1)$ also converges.

  However, even though $1/(2^n - 1)$ *looks like* $1/2^n$, you **can't** use the comparison test because (a) $1/(2^n - 1) > 1/2^n$ and (b) being *large* than a convergent series tells you *nothing!*

  The solution is to use **the limit comparison test** with $a_n = 1/(2^n-1)$ and $b_n = 1/2^n$. Doing so shows that $\lim_{n\to\infty} a_n/b_n = 1$; thus, $\sum a_n$ converges because $\sum b_n$ does.