③ (b) $y'' + x^3 y' + 4x^4 y = 0$

↳ (i) Because $P = 1$ and Q, R both polys, $\frac{Q}{P}$ and $\frac{R}{P}$ have power series for all real #'s. Hence, $(-\infty, \infty)$ are ordinary points (incl. $x_0 = 0$).

(ii) $y = \sum_{n=0}^{\infty} a_n x^n \rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Plug into the ODE:

$$\Rightarrow \quad 0 = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x^3 \sum_{n=1}^{\infty} n a_n x^{n-1} + 4x^4 \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{\boxed{n-2}} + \sum_{n=1}^{\infty} n a_n x^{\boxed{n+2}} + \sum_{n=0}^{\infty} 4a_n x^{\boxed{n+4}}$$

will make all powers "n-2"

$$= \sum_{\boxed{n=2}}^{\infty} n(n-1)a_n x^{n-2} + \sum_{\boxed{n=5}}^{\infty} (n-4)a_{n-4} x^{n-2} + \sum_{\boxed{n=6}}^{\infty} 4a_{n-6} x^{n-2}$$

make all start @ n=6

$$= \Big(\overbrace{2a_2 x^0}^{n=2} + \overbrace{6a_3 x^1}^{n=3} + \overbrace{12a_4 x^2}^{n=4} + \overbrace{20a_5 x^3}^{n=5} + \cancel{30a_6 x^4} + \sum_{n=6}^{\infty} n(n-1)a_n x^{n-2}\Big)$$

don't need

$$+ \Big(\overbrace{1a_1 x^3}^{n=5} + \sum_{n=6}^{\infty} (n-4)a_{n-4} x^{n-2}\Big) + \sum_{n=6}^{\infty} 4a_{n-6} x^{n-2}$$

$$= 2a_2 + 6a_3 x + 12a_4 x^2 + (20a_5 + a_1)x^3 + \sum_{n=6}^{\infty} \left[4a_{n-6} + (n-4)a_{n-4} + n(n-1)a_n\right] x^{n-2}$$

(iii)
- $2a_2 = 0 \rightarrow a_2 = 0$
- $6a_3 = 0 \rightarrow a_3 = 0$
- $12a_4 = 0 \rightarrow a_4 = 0$
- $20a_5 + a_1 = 0 \rightarrow a_5 = \frac{-1}{20} a_1$
- $4a_{n-6} + (n-4)a_{n-4} + n(n-1)a_n = 0, \ n \geq 6$

Using $n=6$:
- $4a_0 + 2a_2 + 30a_6 = 0 \Rightarrow 4a_0 + 0 + 30a_6 = 0 \Rightarrow a_6 = \frac{-4}{30} a_0$

(iv)

a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_0	a_1	0	0	0	$\frac{-1}{20}a_1$	$\frac{-2}{15}a_0$	$\frac{-2}{21}a_1$ (n=7)	0 (n=8)

(v) we have

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + \cdots$$

$$= a_0 + a_1x + 0x^2 + 0x^3 + 0x^4 - \frac{1}{20}a_1x^5 - \frac{2}{15}a_0x^6 - \frac{2}{21}a_1x^7 + 0x^8 + \cdots$$

has a_0

has a_1

$$= a_0\underbrace{\left(1 - \frac{2}{15}x^6 + \cdots\right)}_{y_1} + a_1\underbrace{\left(x - \frac{1}{20}x^5 - \frac{2}{21}x^7 + \cdots\right)}_{y_2}$$

(vi) & (vii)

↳ same as in 3(a).

③ (c) $y''-y'-y=0$

↳ (i) Same as 3(b)

(ii) Plugging $y=\sum_{n=0}^{\infty} a_n x^n$, $y'=\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$, and $y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ into ODE:

$$0=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=1}^{\infty} n a_n x^{n-1}-\sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty} a_n x^n$$

$$=\sum_{n=0}^{\infty} x^n\left[(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}-a_n\right]$$

(iii) $(n+2)(n+1)a_{n+2}-(n+1)a_{n+1}-a_n=0, \quad n\geq 0$

(iv) • $n=0$: $2a_2-a_1-a_0=0 \rightarrow a_2=\frac{1}{2}a_1+\frac{1}{2}a_0$

• $n=1$: $6a_3-2a_2-a_1=0 \rightarrow a_3=\frac{1}{6}(2a_2+a_1)=\frac{1}{6}\left(2\left(\frac{1}{2}a_1+\frac{1}{2}a_0\right)+a_1\right)$

$=\frac{1}{3}a_1+\frac{1}{6}a_0\left[=\frac{1}{6}(2a_1+a_0)\right)$

• $n=2$: $12a_4-3a_3-a_2=0 \rightarrow a_4=\frac{1}{12}(3a_3+a_2)=\frac{1}{12}\left(\frac{1}{2}(2a_1+a_0)+\left(\frac{1}{2}a_1+\frac{1}{2}a_0\right)\right)$

$\frac{1}{8}a_1+\frac{1}{12}a_0\left[=\frac{1}{12}\left(\frac{3}{2}a_1+a_0\right)\right)$

• $n=3$: $20a_5-4a_4-a_3=0 \rightarrow a_5=\frac{1}{20}(4a_4+a_3)=\frac{1}{20}\left(\frac{1}{2}a_1+\frac{1}{3}a_0+\frac{1}{3}a_1+\frac{1}{6}a_0\right)$

$\frac{5}{24}a_1+\frac{1}{40}a_0\left[=\frac{1}{20}\left(\frac{5}{6}a_1+\frac{1}{2}a_0\right)\right)$

• $n=4$: skipped.

a_0	a_1	a_2	a_3	a_4	a_5
a_0	a_1	$\frac{1}{2}a_0+\frac{1}{2}a_1$	$\frac{1}{6}a_0+\frac{1}{3}a_1$	$\frac{1}{12}a_0+\frac{1}{8}a_1$	$\frac{1}{40}a_0+\frac{5}{24}a_1$

3/3

(v) Rewrite

$$
\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\
&= a_0 + a_1 x + \left(\tfrac{1}{2} a_0 + \tfrac{1}{2} a_1\right) x^2 + \left(\tfrac{1}{6} a_0 + \tfrac{1}{3} a_1\right) x^3 + \left(\tfrac{1}{12} a_0 + \tfrac{1}{8} a_1\right) x^4 \\
&\qquad + \left(\tfrac{1}{40} a_0 + \tfrac{5}{24} a_1\right) x^5 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
&= a_0 \underbrace{\left(1 + \tfrac{1}{2} x^2 + \tfrac{1}{6} x^3 + \tfrac{1}{12} x^4 + \tfrac{1}{40} x^5 + \cdots\right)}_{y_1} \\
&\qquad + a_1 \underbrace{\left(x + \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 + \tfrac{1}{8} x^4 + \tfrac{5}{24} x^5 + \cdots\right)}_{y_2}
\end{aligned}
$$

(vi) & (vii)

↳ Same as 3(a) & 3(b).