③ (a) $(x-1)y'' + xy' - y = 0$

↳ (i) $P(x) = x-1 \Rightarrow \frac{Q}{P} = \frac{\text{poly}}{x-1} = \frac{R}{P}$ have power series centered about x_0 for all $x_0 \neq 1$, i.e. $(-\infty, 1) \cup (1, \infty)$. This contains an open interval ~~around~~ around 0, so 0 $\underline{\underline{\text{is}}}$ an ordinary pt.

(ii) Assume $y = \sum_{n=0}^{\infty} a_n x^n \rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \rightarrow y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Plugging in to the ODE:

$$0 = (x-1)y'' + xy' - y = xy'' - y'' + xy' - y$$

$$\Rightarrow 0 = x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

method 1 from preview list

$$= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

(want powers of x to be same, so reindex)

$$= \sum_{n=1}^{\infty} (n+1)\cdot n\, \underset{\uparrow\, a_{n+1}}{} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

want bottom index to match

method 2 from preview list

$$= \sum_{n=1}^{\infty} (n+1)\cdot n\, a_{n+1} x^n - \left(2(1)a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n\right) + \sum_{n=1}^{\infty} n a_n x^n - \left(a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n\right)$$

$$= -2a_2 - a_0 + \sum_{n=1}^{\infty} x^n \left[n(n+1)a_{n+1} - (n+1)(n+2)a_{n+2} + n a_n - a_n\right]$$

(iii) (*) $-2a_2 - a_0 = 0$, and

$n(n+1)a_{n+1} - (n+1)(n+2)a_{n+2} + n a_n - a_n = 0$ for $n \geq 1$ (**)

(iv) From (*), $a_2 = \frac{-1}{2} a_0$. Now, plug in $n = 1, 2, 3, \ldots$ to (**):

- $n=1$: $2a_2 - 6a_3 + a_1 - a_1 = 0 \Rightarrow a_3 = \frac{1}{3} a_2 = \frac{-1}{6} a_0$
- $n=2$: $6a_3 - 12a_4 + 2a_2 - a_2 = 0 \Rightarrow a_4 = \frac{a_2 + 6a_3}{12} = \frac{1}{12}\Big(\underbrace{\tfrac{-1}{2}a_0 - a_0}_{-3/2\, a_0}\Big) = \frac{-1}{8} a_0$
- $n=3$: $12a_4 - 20a_5 + 3a_3 - a_3 = 0 \Rightarrow a_5 = \frac{12a_4 + 2a_3}{20} = \frac{1}{20}\Big(\underbrace{\tfrac{-12}{8}a_0 + \tfrac{-2}{6}a_0}_{-44/24\, a_0}\Big) = \frac{-11}{120} a_0$
- skip $n=4$ b/c messy arithmetic!

a_0	a_1	a_2	a_3	a_4	a_5
a_0	a_1	$\frac{-1}{2} a_0$	$\frac{-1}{6} a_0$	$\frac{-1}{8} a_0$	$\frac{-11}{120} a_0$

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③ [cont'd]

(v) So, our solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$= a_0 + a_1 x + \frac{-1}{2} a_0 x^2 + \frac{-1}{6} a_0 x^3 + \frac{-1}{8} a_0 x^4 + \frac{-11}{120} a_0 x^5 + \cdots$$

has a_1

have a_0

$$= \underbrace{a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{11}{120}x^5 + \cdots\right)}_{y_1} + a_1 \underbrace{x}_{y_2}$$

(vi) $\frac{Q}{P}$ & $\frac{R}{P}$ both have ~~convergent~~ power series about $x_0 = 0$ which converge on $(-\infty, 1)$. Hence, $R.C.(Q/P) = \infty = R.C.(R/P)$, and so

$$\begin{array}{c} R.C.(y_1) \\ \& \\ R.C.(y_2) \end{array} \geq \min\left(R.C.(Q/P), R.C.(R/P)\right)$$

$$= \min(\infty, \infty)$$

$$= \infty.$$

(vii) <u>True</u>. By "the theorem," y_1 & y_2 form a F.S.S., so $W(y_1, y_2)$ is not equal to zero for all x!