1. (a) $y'' + 9y = te^{2t} + t\sin(2t) + 3$

↳ (i) Homogeneous ODE: $y'' + 9y = 0 \longleftrightarrow r^2 + 9 = 0$

$$\longleftrightarrow r = \pm 3i$$

$\Rightarrow$ General solution: $\boxed{y = C_1\cos(3t) + C_2\sin(3t)}$

(ii) $\mathcal{L}(y'' + 9y) = \mathcal{L}(0) \Rightarrow \left(s^2\mathcal{L}(y) - s\,y(0) - y'(0)\right) + 9\mathcal{L}\{y\} = 0$

$$\Rightarrow \mathcal{L}(y)\,(s^2+9) = s\,y(0) + y'(0) \quad (\star)$$

- Now, $y(0) = 2$ & $y'(0) = 2$, so:

$(\star) \Rightarrow \mathcal{L}(y)\,(s^2+9) = 2s + 2$

$$\Rightarrow \mathcal{L}(y) = \frac{2s+2}{s^2+9} = 2\left(\frac{s}{s^2+9}\right) + 2\left(\frac{1}{s^2+9}\right)$$

#6 #5

- Using the table,

$$y = 2\cos(3t) + \frac{2}{3}\sin(3t)$$

(iii) we know $\mathcal{L}\{LHS\} = \mathcal{L}(y)\,(s^2+9) - 2s - 2$; now; for RHS:

- $\mathcal{L}(te^{2t}) = \dfrac{1}{(s-2)^2}$ (# 11)
- $\mathcal{L}(t\sin 2t) = \mathcal{L}\left(-(-t)^{+1}f(t)\right)$ where $f(t) = \sin(2t)$

means 1st derivative

using #19 $\left\{ = -F^{(1)}(s)\right.$, where $F(s) = \mathcal{L}(f(t)) = \dfrac{2}{s^2+4}$

$$= -\left(-2(s^2+4)^{-2}(2s)\right)$$

$$= \frac{4s}{(s^2+4)^2}$$

- $\mathcal{L}(3) = \dfrac{3}{s}$.

Hence, $\mathcal{L}\{LHS\} = \mathcal{L}\{RHS\} \Rightarrow \mathcal{L}(y)(s^2+9) - 2s - 2 = \dfrac{1}{(s-2)^2} + \dfrac{4s}{(s^2+4)^2} + \dfrac{3}{s}$

(iii) [Cont'd]

$$\Rightarrow \mathcal{L}(y) = \frac{\frac{1}{(s-2)^2} + \frac{4s}{(s^2+4)^2} + \frac{3}{s} + 2s + 2}{(s^2+9)}$$

$$= \frac{2s^8 - 6s^7 + 19s^6 - 51s^5 + 72s^4 - 136s^3 + 160s^2 - 48s + 192}{s(s-2)^2(s^2+4)^2(s^2+9)}$$

(b) $y'' + 5y' + 6y = e^t\cos(3t) + t^4$

$\hookrightarrow$ (i) $y'' + 5y' + 6y = 0 \Longleftrightarrow r^2 + 5r + 6 = 0$

$\Longleftrightarrow (r+3)(r+2) = 0$

$\Longleftrightarrow r = -3,\ r = -2$

$\Rightarrow$ Gen. Sol'n: $\boxed{y = c_1 e^{-3t} + c_2 e^{-2t}}$

(ii) $L(y'' + 5y' + 6y) = L(0) \Rightarrow (s^2 L(y) - s\cancelto{2}{y(0)} - \cancelto{2}{y'(0)})$

$+ 5(sL(y) - \cancelto{2}{y(0)})$

$+ 6L(y) = 0$

$\rightarrow s^2 L(y) - 2s - 2 + 5sL(y) - 10 + 6L(y) = 0$

$\Rightarrow L(y)(s^2 + 5s + 6) = 2s + 12$

$\Rightarrow L(y) = \frac{2s+12}{(s+3)(s+2)} \ \underset{\text{fractions}}{\overset{\text{partial}}{=}} \ 8\left(\frac{1}{s+2}\right) - 6\left(\frac{1}{s+3}\right)$

#2 #2

So, using the table:

$y = 8e^{-2t} - 6e^{-3t}$

(iii)

- Know $\mathcal{L}\{LHS\} = \mathcal{L}(y)(s^2+5s+6) - 2s - 12.$
- For $\mathcal{L}\{RHS\}$:
  - $\mathcal{L}(e^t \cos 3t) = \dfrac{s-1}{(s-1)^2+9}$ (by # 10)
  - $\mathcal{L}(t^4) = \dfrac{4!}{s^5} = \dfrac{24}{s^5}$

Hence, $\mathcal{L}\{LHS\} = \mathcal{L}\{RHS\} \Rightarrow \mathcal{L}(y)(s^2+5s+6) - 2s - 12 = \dfrac{s-1}{(s-1)^2+9} + \dfrac{24}{s^5}$

$$\Rightarrow \mathcal{L}(y) = \frac{\frac{s-1}{(s-1)^2+9} + \frac{24}{s^5} + 2s + 12}{s^2+5s+6}$$

$=$ (something we needn't simplify!)

(c) $y'' + 4y' + 4y = e^{-2t} + e^{2t}$

$\hookrightarrow$ (i) $y'' + 4y' + 4y = 0 \longleftrightarrow r^2 + 4r + 4 = 0$

$\longleftrightarrow (r+2)(r+2) = 0$

$\longleftrightarrow r = -2, -2$

$\Rightarrow$ gen soln: $\boxed{y = c_1 e^{-2t} + c_2 t e^{-2t}}$

(ii) $L(y'' + 4y' + 4y) = L(0) \Rightarrow \left(s^2 L(y) - s\cancel{y(0)}^{2} - \cancel{y'(0)}^{2}\right) + 4\left(sL(y) - \cancel{y(0)}^{2}\right) + 4(L(y)) = 0$

$\Rightarrow L(y)(s^2+4s+4) - 2s - 2 - 8 = 0$

NEVERMIND!

$\Rightarrow L(y) = \dfrac{2s+10}{(s+2)^2} \xrightarrow[\text{fractions}]{\text{partial}} \dfrac{6}{(s+2)^2} + \dfrac{2}{s+2}$ ~~[illegible], so you don't know how to finish!~~

$\Rightarrow L(y) = 6te^{-2t} + 2e^{-2t}$

#11 #2

(iii)

- Know: $\mathcal{L}\{LHS\} = \mathcal{L}(y)(s^2+4s+4) - 2s - 10$
- For $\mathcal{L}\{RHS\}$:
  - $\mathcal{L}(e^{-2t}) = \dfrac{1}{s+2}$
  - $\mathcal{L}(e^{2t}) = \dfrac{1}{s-2}$

So, $\mathcal{L}\{LHS\} = \mathcal{L}\{RHS\} \Rightarrow \mathcal{L}(y)(s^2+4s+4) - 2s - 10 = \dfrac{1}{s+2} + \dfrac{1}{s-2}$

$$\Rightarrow \mathcal{L}(y) = \frac{\frac{1}{s+2} + \frac{1}{s-2} + 2s + 10}{(s+2)^2}$$

$$= \frac{2s^3 + 10s^2 - 6s - 40}{(s-2)(s+2)^3}.$$