Example: Determine the interval and radius of convergence for the power series $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{n\left(\ln(n)\right)^{1/2}}$.

SOLUTION:

We use the ratio test with $a_n = \dfrac{x^n}{n\left(\ln(n)\right)^{1/2}}$ and so:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)\left(\ln(n+1)\right)^{1/2}} \cdot \frac{n\left(\ln(n)\right)^{1/2}}{x^n}\right| \tag{1}$$

$$= \left|\frac{x^{n+1}}{x^n} \cdot \frac{n}{n+1} \cdot \frac{\left(\ln(n)\right)^{1/2}}{\left(\ln(n+1)\right)^{1/2}}\right| \tag{2}$$

$$= |x| \cdot \underbrace{\frac{n}{n+1}}_{(a)} \cdot \underbrace{\frac{\left(\ln(n)\right)^{1/2}}{\left(\ln(n+1)\right)^{1/2}}}_{(b)} \tag{3}$$

Note that: For (1), we multiplied by the reciprocal instead of dividing; for (2) we grouped things that looked the same; and for (3), we noticed that the absolute values only affect that x (because all the n things are guaranteed to be positive).

Now, as $n \to \infty$, $(a) \to 1$ and $(b) \to 1$ (which you could get from L'Hopital), so the entire last bit goes to $|x| * 1 * 1 = |x|$ as $n \to \infty$. Call this limit R.

The ratio test says that the series you have converges absolutely if $R < 1$, so you're looking at the interval $R < 1 \iff |x| < 1$. As an interval, this is $(-1, 1)$.

Next, you test the endpoints $x = -1$ and $x = 1$ by plugging those values into the original power series and seeing if the resulting series converges or diverges.

For $x = -1$:

$$\sum_{n=2}^{\infty} \frac{x^n}{n\left(\ln(n)\right)^{1/2}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\left(\ln(n)\right)^{1/2}}.$$

Now, we note that this is an *alternating* series, that the positive part

$$b_n = \frac{1}{n\left(\ln(n)\right)^{1/2}}$$

is decreasing (larger denominator = smaller fraction), and that $b_n \to 0$ as $n \to \infty$. Therefore, by the **alternating series test**, the series *converges* for $x = -1$.

Finally, for $x = 1$:

$$\sum_{n=2}^{\infty} \frac{x^n}{n\left(\ln(n)\right)^{1/2}} = \sum_{n=2}^{\infty} \frac{1}{n\left(\ln(n)\right)^{1/2}},$$

and because the associated function

$$f(x) = \frac{1}{x\left(\ln(x)\right)^{1/2}}$$

is *positive*, *continuous* (for $2 < x < \infty$), and *decreasing* (see above), we can use the **integral test**:

$$\int_2^{\infty} \frac{1}{x\left(\ln(x)\right)^{1/2}}\,dx = 2\sqrt{\ln(x)}\Big|_2^{\infty} = \infty \implies \sum_{n=2}^{\infty} \frac{1}{n\left(\ln(n)\right)^{1/2}} \quad \text{diverges by the integral test.}$$

Therefore, the original power series converges on the interval $(-1, 1)$ (by the ratio test) and at $x = -1$ (by the above), making your final answer:

Interval of convergence: $I = [-1, 1)$; **Radius of convergence:** $R = 1$. □