Exam 4 Preview

Here's a bit of logistical info about the exam.

- There will be 4–5 questions overall, and some will have multiple parts.
- Aside from background sections (§6.1 and §5.1 for backgrounds on Laplace transforms and power series, respectively), the exam will cover the following textbook sections:
  - §6.2 (Laplace transform solutions to ODEs)
  - §5.2 (power series solutions to ODEs/recurrence relations)
  - §5.3 (ordinary/singular points + the Existence Theorem for Power Series Solutions to ODE)
- You should expect the following question formats:
  - computation questions (e.g. solving ODEs from start to finish)
  - multiple-choice questions
  - True/False questions (which may or may not require justification).

  The True/False questions will cover various facts about Laplace transforms (some of which may have come up on Exam 3) and power series solutions to ODE (rich examples of which may come from "the" theorem in §5.3).
- Even though it isn't something we directly covered, you will need to know how to use the following techniques from "old calculus":
  - partial fraction decomposition
  - the ratio test
- There are two essential ways to adjust the indices of series:
  1. Adding to or subtracting from the index. **Note:** If you add to an index, you subtract from the things inside the sum (and vice versa).

     Example: $\sum_{n=1}^{\infty} na_nx^{n-1} \overset{(\star)}{=} \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \overset{(\star\star)}{=} \sum_{n=2}^{\infty}(n-1)a_{n-1}x^{n-2}$.

     To clarify: For ($\star$), we subtracted 1 from the index of the first sum (and thus added 1 to all the n's inside the first sum); for ($\star\star$), we added 2 to the index of the second sum (and thus subtracted 2 from all the n's inside the second sum).
  2. "Peeling off terms" from your sum. Here, we use the property that

     $$\underbrace{b_0+b_1+b_2+b_3+\cdots}_{\sum_{n=0}^{\infty} b_n} = b_0 + \underbrace{(b_1+b_2+b_3+\cdots)}_{\sum_{n=1}^{\infty} b_n} = b_0+b_1+\underbrace{(b_2+b_3+\cdots)}_{\sum_{n=2}^{\infty} b_n}$$

     to rearrange the series.

     Example: $\sum_{n=1}^{\infty} na_nx^{n-1} = \underbrace{1a_1x^0}_{n=1 \text{ term}} + \sum_{n=2}^{\infty} na_nx^{n-1} = \underbrace{1a_1x^0}_{n=1 \text{ term}} + \underbrace{2a_2x^1}_{n=2 \text{ term}} \sum_{n=3}^{\infty} na_nx^{n-1}$.

  Note: Using method 1 changes both the index and what's inside the summation; method 2 only changes the index!

Now, here are some sample questions that you should be able to answer before the exam.

1. For the non-homogeneous ODEs in parts (a), (b), and (c),:
   (i) Find the general solution of the corresponding homogeneous ODE;
   (ii) use Laplace transforms to solve the IVP consisting of the corresponding homogeneous ODE with the initial conditions $y(0) = 2$, $y'(0) = 2$; and
   (iii) use Laplace transforms to find $\mathcal{L}\{y\}$, where y is the solution of the IVP consisting of the given ODE with the initial conditions $y(0) = 2$, $y'(0) = 2$. **Do not "invert" the transform and/or solve for y!**

   (a) $y'' + 9y = te^{2t} + t\sin(2t) + 3$ **Hint**: $t\sin(2t) = -(-t)^1 f(t)$, where $f(t) = \sin(2t)$.

   (b) $y'' + 5y' + 6y = e^t\cos(3t) + t^4$

   (c) $y'' + 4y' + 4y = e^{-2t} + e^{2t}$

2. Indicate whether each of the following questions is True or False.

   (a) If Q and R are polynomials, then $x_0 = 2$ is a singular point for the ODE
   $$(x^2 - 4)y'' + Q(x)y' + R(x)y = 0.$$

   (b) Every function $f(t)$ has a Laplace transform.

   (c) There is no function $f(t)$ whose Laplace transform is $F(s) = \dfrac{1}{2s^2 + 10s + 12}$.

   (d) Every second-order linear homogeneous ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ has a power series solution of the form $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, where $x_0 = 0$.

   (e) Laplace transforms are unique. In other words: If a function $f(t)$ has Laplace transform $F(s)$, then no other function $g(t) \neq f(t)$ may have $F(s)$ as its Laplace transform.

   (f) $\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f(0)$.

   (g) If there is a power series converging to $Q(x)/P(x)$ on the interval $(0, 5/2)$ and a power series converging to $R(x)/P(x)$ on the interval $[-1, 2]$, then $x_0 = 2$ is an ordinary point for the differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$.

   (h) $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f'(0)$.

   (i) If $\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} n(n+1)a_n x^n$ for all x, then $na_n = n(n+1)a_n$ for all n.

   (j) If Q and R are polynomials, then there exists a power series solution to the ODE
   $$(1 - x^2)y'' + Q(x)y' + R(x)y = 0$$
   about the point $x_0 = 0$ whose radius of convergence is $\dfrac{3}{4}$.

3. For the ODEs in parts (a), (b), and (c):

   (i) Find the ordinary points corresponding to the ODE and verify that $x_0 = 0$ is an ordinary point for each;

   (ii) find a power series solution centered at $x_0 = 0$ (but do not solve for the a_n!);

   (iii) write the corresponding equivalence relation(s);

   (iv) write the values for the coefficients a_0, a_1, ..., a_6;

   (v) write your solution from (a) in the form $a_0 y_1 + a_1 y_2$, where y_1 and y_2 are power series solutions about the point x_0;

   (vi) find a lower bound for the radii of convergence for the series y_1 and y_2 found in part (iv); and

   (vii) answer the following: **True or False:** There exists a real number x for which the Wronskian $W(y_1, y_2)$ of the solutions y_1 and y_2 found in part (iv) is nonzero. Justify your answer!

   (a) $(x-1)y'' + xy' - y = 0$

   (b) $y'' + x^3 y' + 4x^4 y = 0$

   (c) $y'' - y' - y = 0$