Exam 2 Preview

Here's a bit of logistical info about the exam.

- There will be 5–7 questions overall, and some will have multiple parts.
- The exam will cover the following textbook sections:
  - §2.6 (exact ODEs)
  - §2.8 (existence/uniqueness theorem)
  - §3.1, §3.3, §3.4 (2nd order linear homogeneous ODEs with constant coefficients)
  - §3.2 (the Wronskian and fundamental systems of solutions)
- You should expect the following question formats:
  - computation questions (e.g. solving ODEs from start to finish)
  - multiple-choice questions
  - True/False questions (which may or may not require justification).

  The True/False questions will mostly look like those from Quiz 2 (on the existence/uniqueness theorem) and won't cover the theoretical aspects of §3.2.

Next, here are the explanations for the True/False questions on Quiz 2. **There will be questions which look very similar to these on Exam 2, despite not having any shown with the "sample problems" on the next page!**

1. The IVP always has a solution if f is continuous in a small rectangle containing x_0. True: f being continuous is enough to conlcude that there's **a** solution; more is needed for the solution to be unique.

2. The IVP always has a *unique* solution if f is continuous in a small rectangle containing x_0. False: f being continuous is enough to conlcude that there's **a** solution; more is needed for the solution to be unique.

3. The IVP always has a *unique* solution if $\partial f/\partial y$ is continuous in a small rectangle containing x_0. False: If f were also continuous, this would be true; however, we can't conclude anything about the continuity of f based ont he continuity of $\partial f/\partial y$. For example: Imagine that f is a function with **no** y**'s** which is *also* discontinuous: Then $\partial f/\partial y = 0$ is continuous everywhere but f is discontinuous.

4. The IVP always has a solution if f and $\partial f/\partial x$ are both continuous in a small rectangle containing x_0. True: f being continuous is enough to conlcude that there's **a** solution; more is needed for the solution to be unique.

5. The IVP always has a *unique* solution if f and $\partial f/\partial x$ are both continuous in a small rectangle containing x_0. False: This looks a lot like the existence and uniqueness theorem, except **that** theorem involves $\partial f/\partial y$ being continuous, not $\partial f/\partial x$. A counterexample to this was done in class: We wrote down three solutions to the ODE $y' = y^{1/3}$ in class despite $f = y^{1/3}$ and $\partial f/\partial x = 0$ are both continuous everhwhere.

6. The IVP may have multiple solutions. True: The autonomous ODEs we studied in §2.5 had multiple solutions (e.g. multiple equilibrium solutions).

7. The IVP may have no solution. True: This could happen in lots of situations, but one easily-imagined one is that the initial value given can't be plugged in to the solution. For example: If I had $y' = 1/x, x > 0$ as an ODE, then we can separate and integrate to get $y = \ln(x) + C$ as a general solution. If we turn this into an IVP with the initial condition $y(-5) = 2$, however, we'd have something that doesn't exist since we can't plug $x = -5$ into $y = \ln(x) + C$.

8. If the IVP has a unique solution, the existence and uniqueness theorem tells you that the solution is valid on an x-interval containing x_0. True: This is the only thing that theorem tells you about the solution...

9. If the IVP has a unique solution, the existence and uniqueness theorem helps you find the x-interval containing x_0 on which the solution is valid. False: ...and this is one of the many things that theorem **doesn't** tell you! Remember: The existence and uniqueness theorem tells you that if f and $\partial f/\partial y$ are both contiuous in a rectangle containing (x_0, y_0), then the IVP has a unique solution defined on an x-interval containing x_0; it **doesn't** tell you *which* x-interval!

10. If $f(x, y) = 0$, then the IVP has a unique solution. True: Here, $f = 0$ and $\partial f/\partial y = 0$ are both continuous everywhere, so you can use the existence and uniqueness theorem. Alternatively, you can also solve this IVP: The general solution would be $y = C$, and using the initial value $y(x_0) = y_0$ tells you that $C = y_0$, and hence that the (unique) particular solution is the constant function $y = y_0$.

Finally, here are some sample questions that you should be able to answer before the exam.

1. Solve the IVP
$$\sin y + (x\cos y + 3y^2)y' = -2x, \quad y(0) = \pi.$$
Note: This is the same ODE that was on both Exam 1 *and* Quiz 2; the one on Exam 2 will be different, but at this point, there are still lots of people who can't solve this one! If you *can* solve this one, focus on problems 1–15 in §2.6 as alternatives.

2. The following ODEs are not exact. For each, find an integrating factor which makes it exact.
   (a) $(xy - \cos y)\, y' - 4 = 0$
   (b) $\left(4x^2y + 2xy + \frac{4}{3}y^3\right) + (x^2 + y^2)y' = 0$

3. For which of the following initial conditions does the IVP
$$(\tan y)\,\frac{dy}{dx} + \ln x = \ln(\ln(3x)), \quad y(x_0) = y_0$$
have a unique solution? **There may be more than one!**
   i. $y\left(\frac{1}{3}\right) = 2$
   ii. $y\left(\frac{1}{6}\right) = 2$
   iii. $y\left(\frac{1}{6}\right) = \frac{\pi}{2}$
   iv. $y\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$
   v. $y\left(\frac{e}{3}\right) = 0$
   vi. $y\left(\frac{e}{3}\right) = \frac{3\pi}{2}$
   vii. $y\left(\frac{\pi}{2}\right) = \frac{3e}{2}$
   viii. $y\left(-\frac{e}{3}\right) = \frac{\pi}{4}$
   ix. None of These

4. Solve the following IVPs.
   (a) $y'' - 4y' + 53y = 0,\ y(\pi) = \pi,\ y'(\pi) = -\pi$
   (b) $4y'' - 3y' - 7y = 0,\ y(0) = 0,\ y'(0) = 3$
   (c) $y'' + 2y' + y = 0,\ y(1) = 2,\ y'(1) = -1$

5. For each of the following functions y, determine a second-order linear homoogeneous ODE having y as a general solution.
   (a) $y = c_1e^{2x} + c_2xe^{2x}$
   (b) $y = e^{-3x}\left(c_1\cos 2x + c_2\sin 2x\right)$
   (c) $y = c_1e^{-5x} + c_2e^{5x}$

6. Let $y_1 = e^x \sin 3x$ and $y_2 = e^x \cos 3x$.

(a) Verify that y_1 and y_2 are solutions to the ODE $y'' - 2y' + 10y = 0$.

(b) Find the Wronskian $W(y_1, y_2)$.

(c) Do the functions y_1 and y_2 constitute a fundamental set of solutions for the ODE $y''-2y'+10y = 0$? Why or why not?

(d) **True or False:** There exists at least one solution y_3 to the ODE $y'' - 2y' + 10y = 0$ such that $y_3 \neq c_1y_1 + c_2y_2$ for any choice of constants c_1 and c_2. **Justify your claim!**