Autonomous ODEs & Population Dynamics

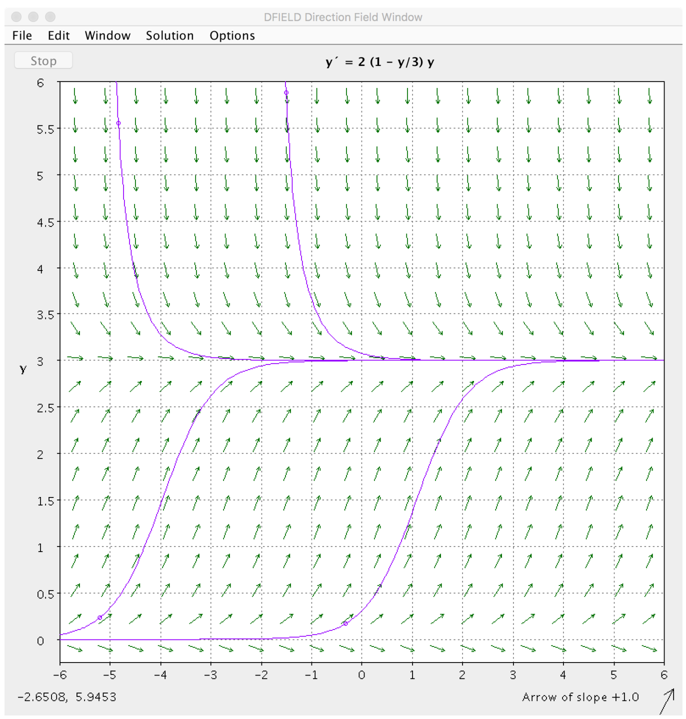
In class, it was very obvious that you guys were either unhappy with the lecture on Monday or a bit confused by it...or both! So, for the sake of clarity, I'm going to break things down a bit more in this handout *and* do another (somewhat harder!) example that we don't have time to cover in class.

The Main Ideas: *What are we even* doing *right now anyway?!*

Main, Big-Picture Goal

The main, big-picture goal at this point is to sketch (on an xy-plane) solutions/integral curves of an *autonomous ODE* (defined below) without actually solving the ODE!

We also want to develop the tools needed to do this *without* having to graph the corresponding slope field, because let's face it: Slope fields are tedious (at best) to create and are difficult (on average) to utilize correctly! If you don't believe this, compare the "correct solution" to problem 3 on HW1 with what your initial approximate sketches looked like!



There's software like **`DFIELD`** *which lets us graph solutions of ODEs without solving them. The goal of this section is to get equally-nice pictures sans slope field!*

Intermediate Goal which will Deliver Us to the Main, Big-Picture Goal

In order to get to the xy-plane/integral curves we want, we're first going to correctly sketch and label the *phase line* (defined below) corresponding to the given autonomous ODE.

As shown in class, one of the main tools we're going to use to get *there* is a "y-versus-$f(y)$ graph" that we're *also* going to construct!

How do I know what steps to do when in this nonsense process?!

Good question!

While it's not entirely possible to write down a precise recipe for what works for *every* single problem, here's a list that attempts to summarize things somewhat logically (and in-order!).

- Find the *equilibrium solutions* of the ODE. By definition, these will be solutions of the form $y = \text{const}$.
  - On the **"y-versus-$f(y)$ graph"**, these will be points/circles at the values *const*, on the y-axis.
  - On the **phase line**, these will be points/circles at the values const.
  - On the **final xy-plane graph**, these will be horizontal lines of the form $y = \text{const}$.
- Find the *local extrema* of the ODE and use these to graph the curve $f(y)$ on the "y-versus-$f(y)$ graph." By definition, these will be points where $f'(y) = 0$ (see the second part of the below boxed **note**).
  - On the **"y-versus-$f(y)$ graph"**, these will be points/circles at the local maxes and local mins.
  - On the **phase line**, these will be vertical tick marks at the values y for which $f'(y) = 0$.
  - On the **final xy-plane graph**, these will be y-values marking where the solution curves change concavity.
- Find where y (as a function of x) is *increasing* and *decreasing*.
  - On the **"y-versus-$f(y)$ graph"**, these will be regions where the graph of $f(y)$ is positive (for increasing) and negative (for decreasing).
  - On the **phase line**, these will be arrows pointing right (for increasing) and pointing left (for decreasing).
  - On the **final xy-plane graph**, these will be y-values marking where the solution curves are increasing and decreasing.
- Find where y (as a function of x) is *concave up* and *concave down*.
  - On the **"y-versus-$f(y)$ graph"**, these will be regions where $f(y)$ and $f'(y)$ have the same signs (for concave up) and where $f(y)$ and $f'(y)$ have the opposite signs (for concave down).
  - **You don't put these on the phase line!**
  - On the **final xy-plane graph**, these will be y-values marking where the solution curves are concave up and concave down.
- Find the remaining points where y (as a function of x) has *inflection points*.

  Note: You already found some of these when you found the local extrema of the ODE!
  - On the **"y-versus-$f(y)$ graph"**, these will be some (**but not all!**) of the points where $f(y)$ has a local max/min or crosses the y-axis.
  - **If you put these on the phase line** (and you don't have to), they can appear as vertical tick marks.
  - On the **final xy-plane graph**, these will be y-values marking where the solution curves change concavity.

Definitions and Necessary Concepts

Definition 1. An ODE is _autonomous_ if it has the form $\frac{dy}{dx} = f(y)$ for some function f (in which x does not appear!).

Ex. $\frac{dy}{dx} = y$, $\frac{dy}{dx} = 2y$, $\dots$, $\frac{dy}{dx} = ny$ ($n = \text{const}$).

Ex. $\frac{dy}{dx} = y^2$, $\frac{dy}{dx} = y^3$, $\dots$, $\frac{dy}{dx} = y^n$ ($n = \text{const}$).

Ex. $\frac{dy}{dx} = e^y \sin y \cos(\cos(\tan y)) + y \sin e^{-y^2} - y^{y^{y^y}}$

Note:

1. As shown in class, every autonomous ODE is separable (but not vice versa!); this means that we *can* (and sometimes *will*) solve these ODEs explicitly!
2. Because $dy/dx = f(y)$ and because $y = y(x)$ is a function of x, questions requiring the *second* derivative d^2y/dx^2 of y need the chain rule:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx} f(y(x)) = f'(y(x))\, y'(x) = f'(y)f(y).$$

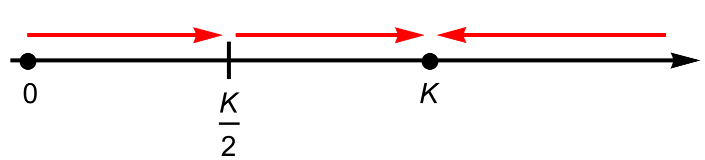
This verifies the above claim: In order to be concave up (concave down), we need that d^2y/dx^2 is positive (negative), and $f'(y)f(y)$ is positive (negative) if and only if $f(y)$ and $f'(y)$ have the same (opposite) signs!

Definition 2. The _phase line_ corresponding to an autonomous ODE is a copy of the y-axis with information about the equilibrium solutions *and* the increasing/decreasing behavior of y encoded as above.

Ex. Let $r > 0$ and $K > 0$ be constants. As we saw in class,

$$\frac{dy}{dx} = r\left(1 - \frac{y}{K}\right) y$$

is an autonomous ODE representing logistic growth, and its corresponding phase line is:



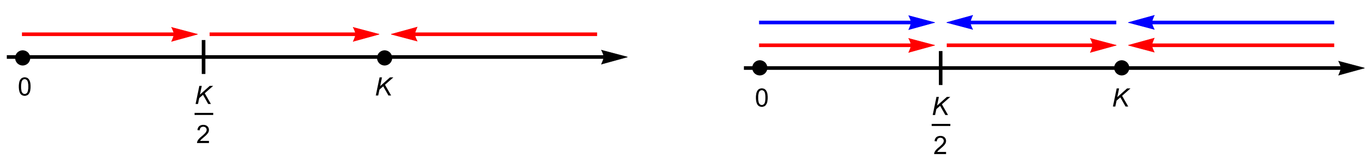
Mathematically, this tells us that the equilibrium solutions for that ODE are $y = 0$ and $y = K$, that there is an inflection point at $y = K/2$, and that the solutions must be *increasing* for y in $(0, K)$ and *decreasing* for y in (K, ∞).

> **Note:**
> We got the data for this phase line by looking at the features of the "y-versus-$f(y)$ graph" for the ODE. The qualitative analysis used on that graph doesn't tell us *all* the important features possessed by our solutions, however: For instance, we *also* have an inflection point at $y = K$! This point would likely go unnoticed unless you also use other methods of analysis (e.g. studying the derivatives of $f(y)$).

For the sake of convenience, we sometimes add more to the phase line than the bare minimum needed to describe the data mentioned so far. In the example from class,

$$\frac{dy}{dx} = \underbrace{r\left(1 - \frac{y}{K}\right)y}_{f(y)}$$

and in addition to the intervals where $f(y) > 0$ and $f(y) < 0$ (shown with red arrows above), we can *also* add in information about where $f'(y) > 0$ and $f'(y) < 0$:



On the left is the "standard phase line" discussed above; a modified version of this phase line is on the right. The blue arrows show where $f'(y) > 0$ and $f'(y) < 0$ with rightward- and leftward-facing arrows, respectively. This modified version will be used throughout.

And now, one last (pair of) definition(s):

Definition 3. An equilibrium solution $y =$ const to an ODE is *asymptotically stable* if all solutions on both sides of the line $y =$ const (in the xy-plane) limit/converge to const as $x \to \infty$.

An equilibrium solution $y =$ const to an ODE is *asymptotically unstable* if no solutions on either side of the line $y =$ const (in the xy-plane) limit/converge to const as $x \to \infty$. In particular, the equilibrium solution $y =$ const is asymptotically unstable if and only if the only solution to the ODE which "always stays close" to $y =$ const as $x \to \infty$ is $y =$ const itself.

There are other related notions, too, like "semistable solutions," etc.; before saying too many more words, however, I think it's time to shift to some examples!

Example 1

In this example, we're going to consider the autonomous ODE

$$\frac{dy}{dx} = \underbrace{-r\left(1 - \frac{y}{T}\right)y}_{f(y)}. \tag{1}$$

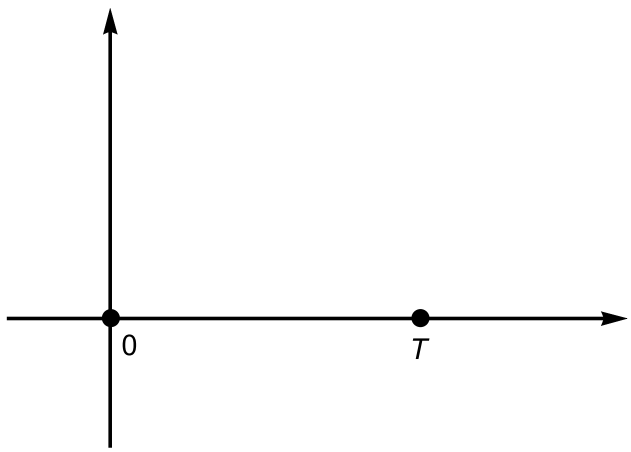
Here, $r > 0$ and $T > 0$ are positive constants. To graph the solutions of (1), we're going to break the process into steps as described above.

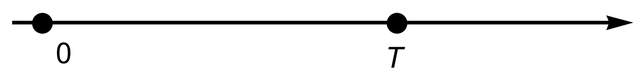
Equilibrium solutions:

Recall that an *equilibrium solution* is a solution of the form $y = \text{const}$ so that $\frac{dy}{dx} = 0$. In this case,

$$\frac{dy}{dx} = 0 \iff -r = 0 \ \underline{\text{or}} \ 1 - \frac{y}{T} = 0 \ \underline{\text{or}} \ y = 0; \tag{2}$$

because $r \neq 0$ is assumed, the equilibrium solutions are $y = T$ and $y = 0$. We plot those on the "y-versus-$f(y)$ graph" and on the phase line.





On the left, the "y-versus-f(y) graph"; the phase line is on the right. We'll continue to post updated versions as we go.

Sketch the "y-versus-$f(y)$" curve:

By manipulating equation (1), we see that

$$f(y) = -ry + \frac{r}{T}y^2$$

has a graph which is an upward-facing parabola with respect to y. Before proceeding, we want to sketch what that looks like on the "y-versus-$f(y)$" graph.

In order to do this step, we need to find the vertex (or *local min*) of the parabola. To do this, we <u>can</u> use the formula

$$\text{vertex} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right),$$

but for later examples, what we'll see is that finding the *critical points* for $f(y)$ (i.e., where $f'(y) = 0$) will work in the most general cases.

Using the second part of the above $\boxed{\text{boxed}}$ **note**, it follows that

$$\frac{d^2y}{dx^2} = f(y)f'(y) = \underbrace{\left(-r\left(1 - \frac{y}{T}\right)y\right)}_{f(y)} \underbrace{\left(-r + \frac{2r}{T}y\right)}_{f'(y)}, \tag{3}$$

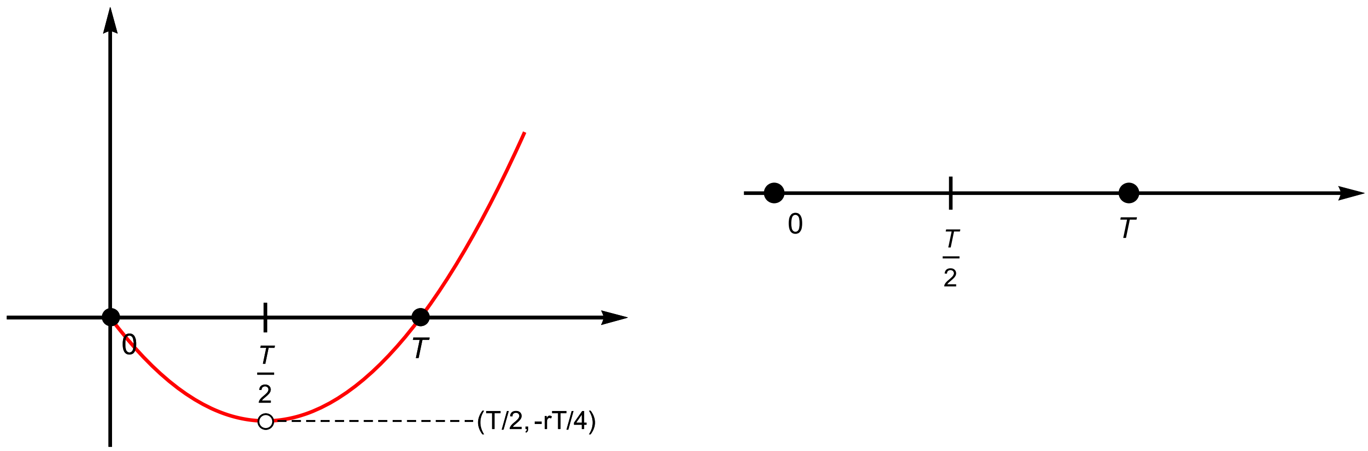
and the critical points for f will be where $f(y) = 0$ and $f'(y) = 0$. In particular, $f(y) = 0$ yields the critical points already found in (2); $f'(y) = 0$ yields one new critical point, namely

$$y = \frac{T}{2}.$$

Plugging into f, we see that the vertex of the parabola is

$$\left(\frac{T}{2}, f\left(\frac{T}{2}\right)\right) = \left(\frac{T}{2}, \frac{-rT}{4}\right), \tag{4}$$

and now, we represent the data in (4) on both the "y-versus-$f(y)$ graph" and the phase line:



On the left is the "y-versus-$f(y)$ graph" featuring the newly-added vertex; the corresponding phase line is on the right.

Determine where y is increasing and decreasing:

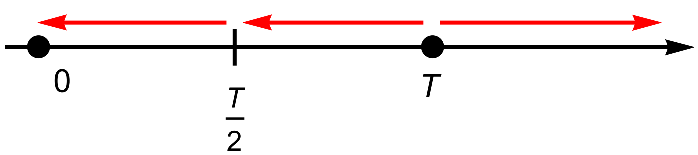
By definition,

- y is increasing as a function of x when the "y-versus-$f(y)$" curve is above the y-axis; and
- y is decreasing as a function of x when the "y-versus-$f(y)$" curve is below the y-axis.

Based on the above figure, the "y-versus-$f(y)$" curve is below the y-axis for $0 < y < T$ and above the y-axis for $y > T$. Thus,

$$y \text{ is increasing on } (T, \infty) \text{ and } y \text{ is decreasing on } (0, T).$$

Now, we encode that data on the phase line by putting a **rightward** arrow on intervals where y is increasing and a **leftward** arrow where y is decreasing.



Encoding increasing/decreasing information on the phase line.

Determine where y is concave up and concave down:

In particular, we want to figure out where $\frac{d^2y}{dx^2} > 0$ and $\frac{d^2y}{dx^2} < 0$, and to do so, we refer back to equation (3) and notice that

$$\frac{d^2y}{dx^2} = f(y)f'(y) > 0 \iff f(y) \text{ and } f'(y) \text{ have the same sign;}$$

similarly,

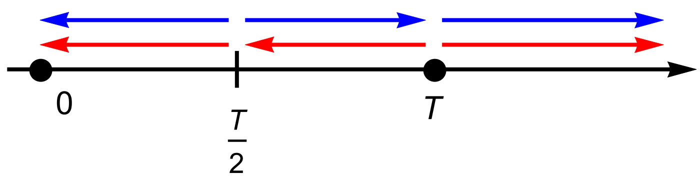
$$\frac{d^2y}{dx^2} < 0$$

whenever $f(y)$ and $f'(y)$ have opposite signs.

By determining where y is increasing and decreasing, we've figured out where $f(y) > 0$ and $f(y) < 0$, respectively. Therefore, we only need to examine the sign of

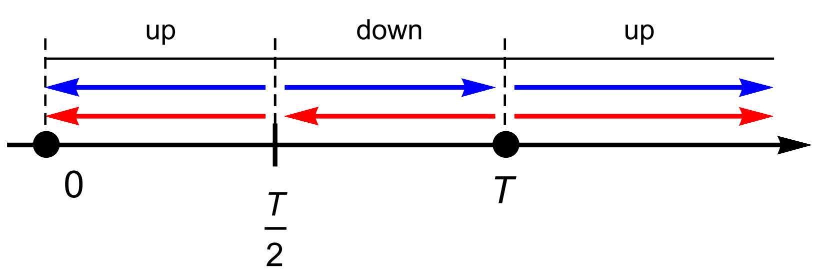
$$f'(y) = -r + \frac{2r}{T}y,$$

and we see[1] that $f'(y) > 0 \iff -r + \frac{2r}{T}y > 0 \iff y > \frac{T}{2}$. We combine this data with the phase line as follows[2]:



*Encoding on the phase line where $f(y) > 0$, $f(y) < 0$ **and** where $f'(y) > 0$, $f'(y) < 0$. As before, the red rightward and leftward arrows indicate where $f(y) > 0$ and $f(y) < 0$, respectively (i.e. where y is increasing and decreasing, respectively). A new addition is the blue rightward and leftward arrows, indicating intervals where $f'(y) > 0$ and $f'(y) < 0$, respectively.*

Based on the dialogue above, y is concave up on intervals where the red and blue arrows are pointing in the same direction and concave down when the red and blue arrows are pointing in opposite directions. We indicate this as follows:



Encoding concavity on the phase line. As stated, y is concave up when f and f' have the same sign (i.e. when the red and blue arrows are pointing in the same direction); it's concave down otherwise.

As **intervals**, we have:

$$y \text{ is concave } \underline{\text{up}} \text{ on } \left(0, \frac{T}{2}\right) \cup (T, \infty) \text{ and } y \text{ is concave } \underline{\text{down}} \text{ on } \left(\frac{T}{2}, T\right).$$

Determine the inflection points of y:

After filling in the last phase line, this data is given for free:

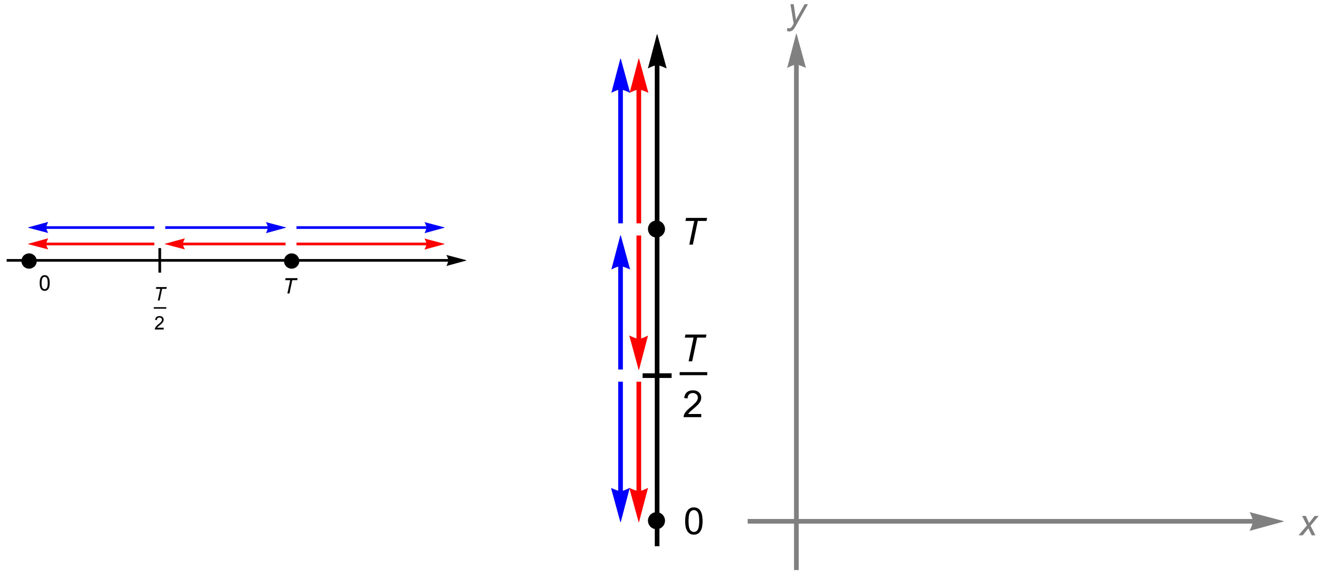
$$y \text{ has inflection points when } y = \frac{T}{2} \text{ and when} y = T.$$

[1] You can also just look at where the graph of $f(y)$ is increasing/decreasing. These notions are equivalent.

[2] This isn't something you have to do: I do it here for clearer exposition.

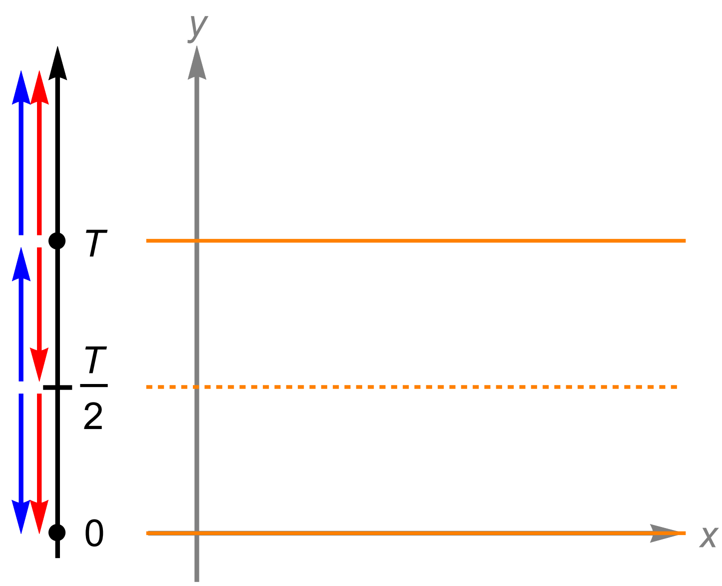
Turning phase lines into drawn graphs

First, we rotate the phase line so that it's vertical and we place it next to an empty xy-plane:



The phase line (on the left) is flipped and put next to an empty xy-plane.

Next, we plot the equilibrium solutions $y = 0$ and $y = T$ on the xy-plane; I'll also add in a dotted line corresponding to the inflection point $y = T/2$, though this is optional.

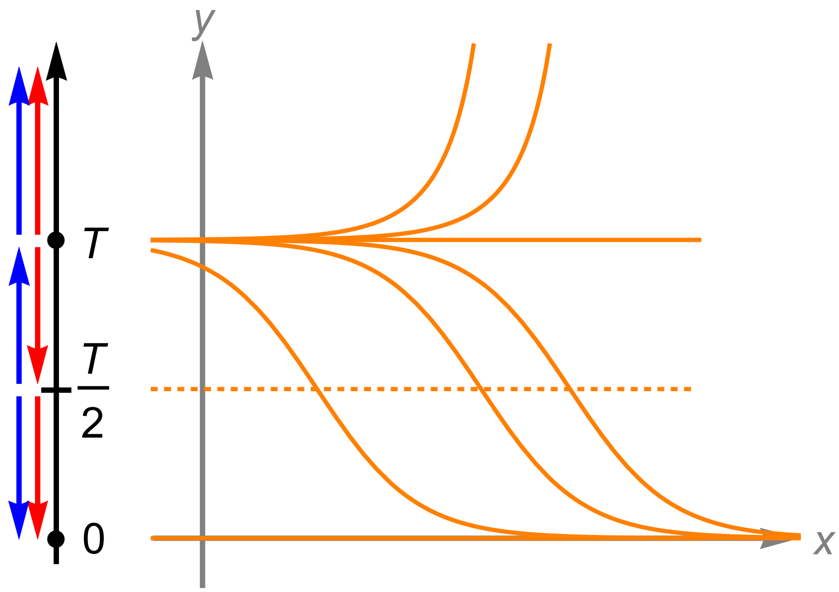


The equilibrium solutions are added (in orange).

The data we have on solutions y is as follows:

- y should be decreasing and concave up on $(0, T/2)$;
- y should be decreasing and concave down on $(T/2, T)$; and
- y should be increasing and concave up on (T, ∞).

That means that y should look like exponential growth on the interval (T, ∞) and should look like the flipped version of the "logistic growth elongated S-shape" on $(0, T)$:

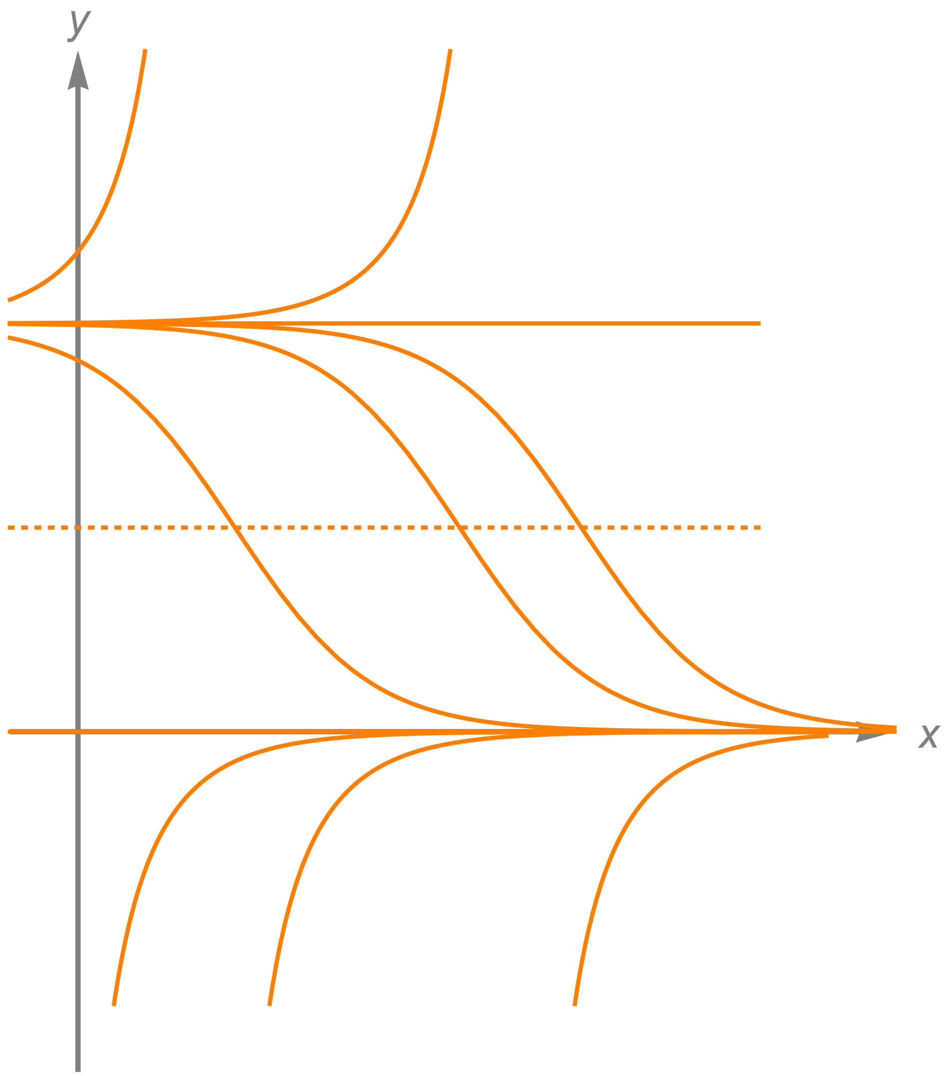


A sketch of various solutions (in orange) to the ODE $\dfrac{dy}{dx} = -r\left(1 - \dfrac{y}{T}\right)y$.

Conclusion

Based on the above graphical information, we can say that

- $y = T$ is **asymptotically unstable**, as the only solution that "stays near $y = T$ for large x" is $y = T$ itself;
- $y = 0$ is **asymptotically stable**. We can see this by extending the model to allow for solutions $y < 0$ (see graph on next page), in which case we see that all solutions in the intervals $0 < y < T$ and $y < 0$ tend towards 0 as $x \to \infty$.



A sketch of various solutions (in orange) to the ODE $\frac{dy}{dx} = -r\left(1 - \frac{y}{T}\right)y$ *where here, we allow* $y < 0$ *to occur. Notice that all solutions in the intervals* $0 < y < T$ *and* $y < 0$ *tend towards* 0 *as* $x \to \infty$.

Example 2

In this example, we're going to consider the somewhat more difficult autonomous ODE

$$\frac{dy}{dx} = \underbrace{-r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y}_{f(y)}. \tag{5}$$

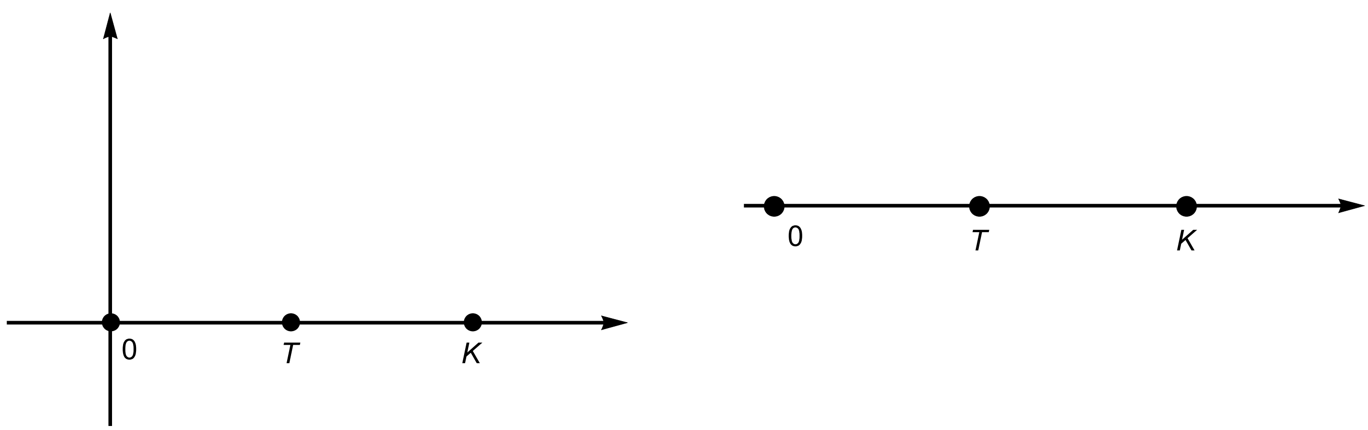
Here, we assume that r, T, and K are positive constants; this time, we further assume that $0 < T < K$. As above, we're going to graph the solutions of (5) by breaking the process into the appropriate steps.

Equilibrium solutions:

We again consider the equation $\frac{dy}{dx} = 0$, and here,

$$\frac{dy}{dx} = 0 \iff -r = 0 \;\;\underline{\text{or}}\;\; 1-\frac{y}{T} = 0 \;\;\underline{\text{or}}\;\; 1-\frac{y}{K} = 0 \;\;\underline{\text{or}}\;\; y = 0.$$

Again, $r \neq 0$, so this time, there are *three* equilibrium solutions: $y = T$, $y = K$, and $y = 0$. As above, we plot those on the "y-versus-$f(y)$ graph" and on the phase line.



On the left, the "y-versus-$f(y)$ graph"; the phase line is on the right. Note that we ***assumed*** *that $0 < T < K$; otherwise, we wouldn't know whether T or K was leftmost.*

Sketch the "y-versus-$f(y)$" curve:

From equation (5),

$$f(y) = -\frac{ry^3}{KT} + \frac{ry^2}{K} + \frac{ry^2}{T} - ry,$$

and on the "y-versus-$f(y)$ graph," this curve is a cubic whose graph approaches $+\infty$ as $y \to -\infty$ and $-\infty$ as $y \to +\infty$; in particular, it starts at the "top left" and is directed towards the "bottom right." We want to fill in the distinguishing features of the graph before proceeding.

As before, we'll use the second part of the above boxed **note**:

$$\frac{d^2y}{dx^2} = f(y)f'(y) = \underbrace{\left(-r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y\right)}_{f(y)}\underbrace{\left(-\frac{3ry^2}{KT} + \frac{2ry}{K} + \frac{2ry}{T} - r\right)}_{f'(y)}. \tag{6}$$

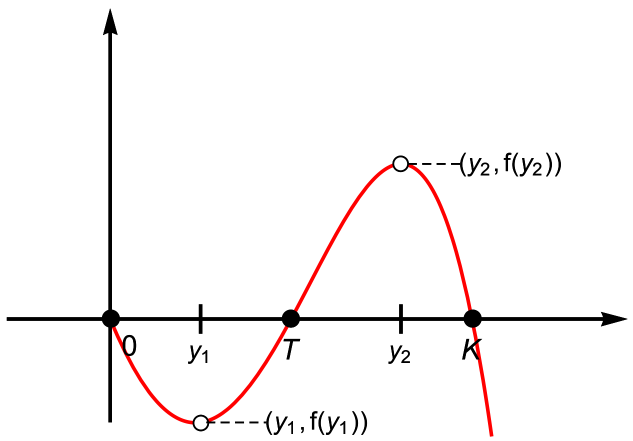
Once again, we identify the new critical points to be the values y for which $f'(y) = 0$, namely those y for which

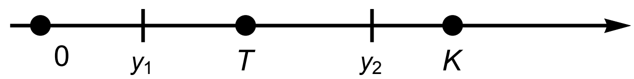
$$-\frac{3ry^2}{KT} + \frac{2ry}{K} + \frac{2ry}{T} - r = 0; \tag{7}$$

solving this equation requires the quadratic formula, but doing so yields two solutions y_1 and y_2 of the form

$$y_1 = \frac{1}{3}\left(K + T - \sqrt{K^2 - KT + T^2}\right) \quad \text{and} \quad y_2 = \frac{1}{3}\left(K + T + \sqrt{K^2 - KT + T^2}\right).$$

We won't worry about plugging these into f to find $f(y_1)$ or $f(y_2)$; instead, we note simply that $f(y_1) < 0$ and $f(y_2) > 0$. This is enough to represent the data on both the "y-versus-$f(y)$ graph" and the phase line:





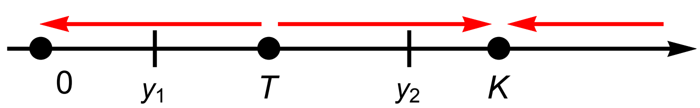
On the left is the "y-versus-$f(y)$ graph" featuring the newly-added local extrema; the corresponding phase line is on the right.

Determine where y is increasing and decreasing:

As in the first example, y is increasing (decreasing) as a function of x when the "y-versus-$f(y)$" curve is above (below) the y-axis.

Based on the above figure, the "y-versus-$f(y)$" curve is above the y-axis for $T < y < K$ and below the y-axis elsewhere (for $0 < y < T$ and for $y > K$). Thus, y is **increasing** on the interval (T, K) and is **decreasing** on the set $(0, T) \cup (K, \infty)$.

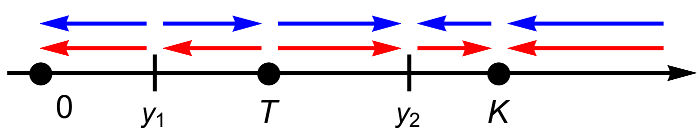
This is shown on the following phase line.



Encoding increasing/decreasing information on the phase line.

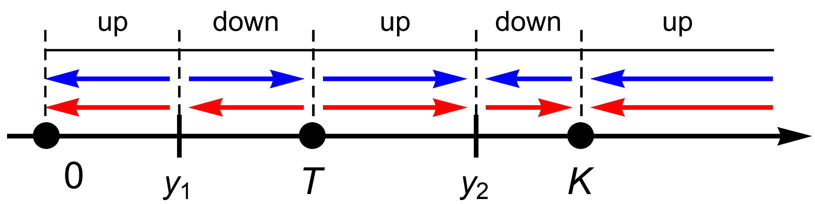
Determine where y is concave up and concave down:

Looking at the "y-versus-$f(y)$" curve, we see that $f(y)$ is increasing on the interval (y_1, y_2) and is decreasing on $(0, y_1) \cup (y_2, \infty)$. We can put this on the phase line using blue arrows:



Encoding on the phase line where $f(y) > 0$, $f(y) < 0$ ***and*** *where $f'(y) > 0$, $f'(y) < 0$ with red and blue arrows, respectively.*

And now, we make the concavity labeling explicit



Encoding concavity on the phase line. As stated, y is concave up when f and f' have the same sign (i.e. when the red and blue arrows are pointing in the same direction); it's concave down otherwise.

As **intervals**, we have:

$$y \text{ is concave } \underline{\text{up}} \text{ on } (0, y_1) \cup (T, y_2) \cup (K, \infty) \text{ and } y \text{ is concave } \underline{\text{down}} \text{ on } (y_1, T) \cup (y_2, K).$$

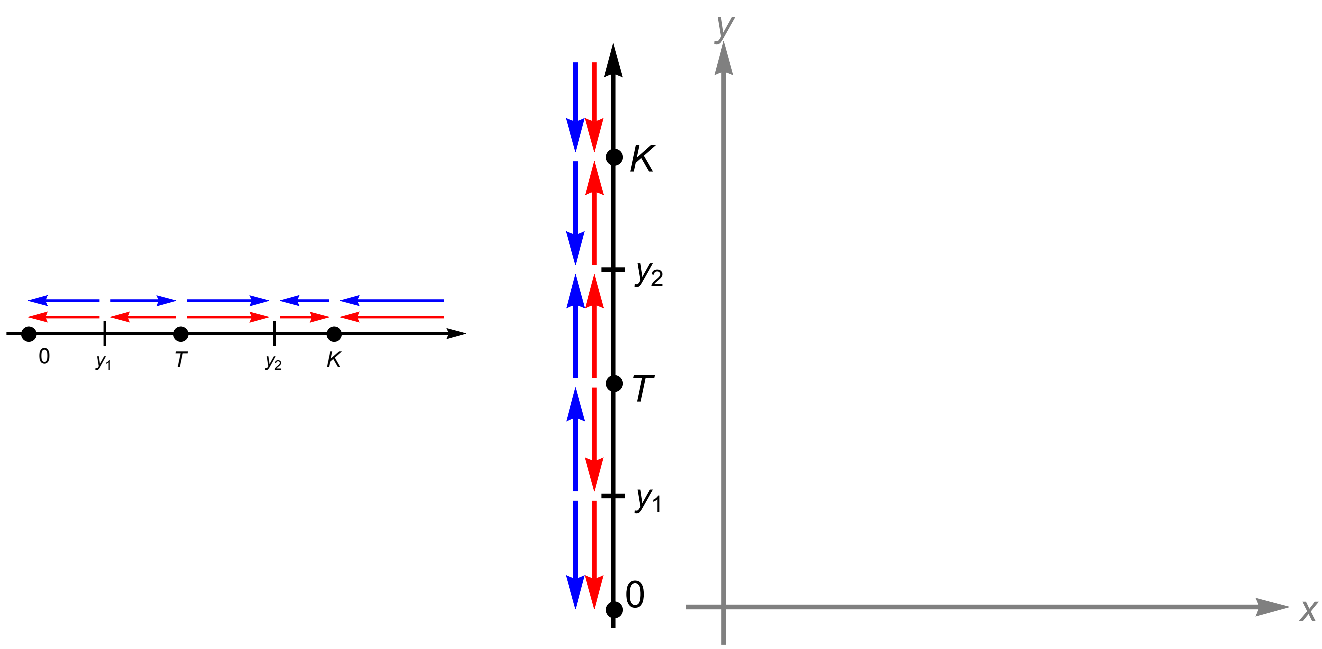
Determine the inflection points of y:

After filling in the last phase line, this data is given for free:

$$y \text{ has inflection points when } y = y_1, y = T, y = y_2, \text{ and } y = K.$$

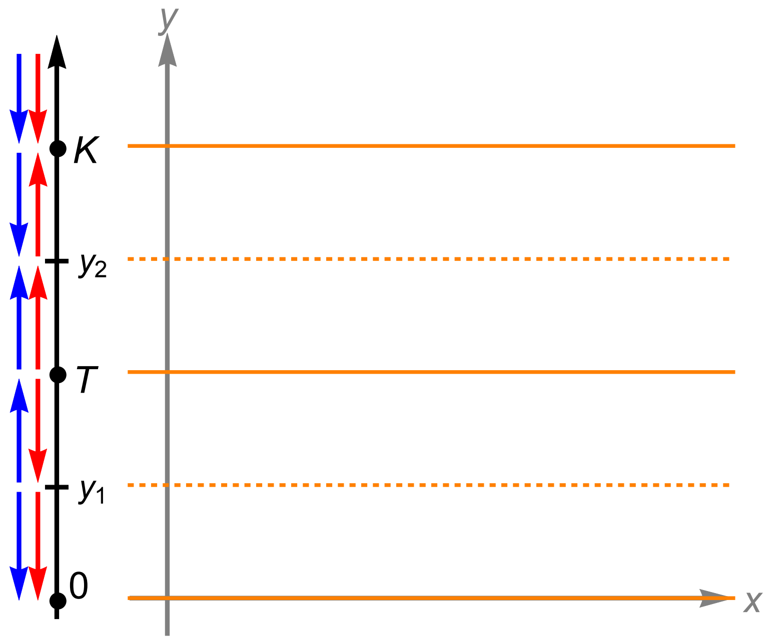
Turning phase lines into drawn graphs

As above, we rotate the phase line so that it's vertical and we place it next to an empty xy-plane:



The phase line (on the left) is flipped and put next to an empty xy-plane.

Next, we plot the equilibrium solutions $y = 0$, $y = T$, and $y = K$ on the xy-plane; as above, I've also added a(n optional) dotted line corresponding to the inflection points $y = y_1$ and $y = y_2$.



The equilibrium solutions are added (in orange).

The data we have on solutions y is as follows:

- y should be decreasing and concave up on $(0, y_1)$;
- y should be increasing and concave down on (y_1, T);
- y should be increasing and concave up on (T, y_2);
- y should be decreasing and concave down on (y_2, K); and
- y should be decreasing and concave up on (K, ∞);

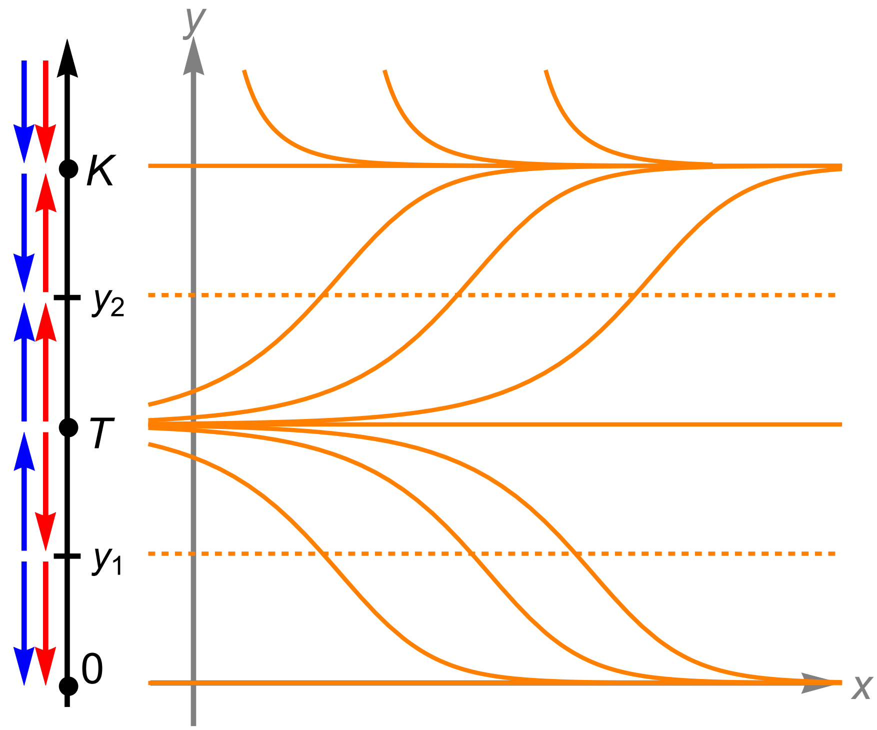
That means that y should look like exponential decay on the interval (K, ∞), like the "logistic growth elongated S-shape" on (T, K), and like the flipped version of the "logistic growth elongated S-shape" on $(0, T)$.

The graph (on the next page) matches this intuition.

Conclusion

As above, we can extend the ODE to allow for solutions $y < 0$ (not pictured), and doing so would show that

- $y = K$ is **asymptotically stable**;
- $y = T$ is **asymptotically unstable**; and
- $y = 0$ is **asymptotically stable**.



A sketch of various solutions (in orange) to the ODE $\dfrac{dy}{dx} = -r\left(1 - \dfrac{y}{T}\right)\left(1 - \dfrac{y}{K}\right)y.$