Exam 3

Jul 17, 2017

MAP 2302—ODE, SUMMER 2017

(CLEARLY!) PRINT NAME: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **5 problems** and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by circling them and/or drawing a box around them.
   - More than one choice may be a correct answer for a question; if so, circle all correct answers!
   - There may be correct answers which aren't listed; in this case, only focus on the choices provided!
8. The notation $y^{(n)}$ **always** means "the n^{th} derivative of y", e.g. $y^{(4)}$ is the fourth derivative of y and is equivalent to y'''' (with 4 primes).
9. i **always** denotes the imaginary unit: $i = \sqrt{-1}$, i.e. $i^2 = -1$.
10. $F(s)$ **always** denotes the Laplace transform of the function $f(t)$, i.e. $F(s) \overset{\text{def}}{=} \mathcal{L}\{f(t)\}$.

Question	1 (20)	2 (20)	3 (15)	4 (30)	5 (15)	Total (100)
Points						

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*5 pts ea.*) Consider the second-order linear IVP

$$(x-3)(2x+1)y'' + (\ln x)y' - \left(\frac{3}{(x-e)\ln x}\right)y = \frac{9\sqrt{x+1}}{6x-4}, \quad y(x_0) = y_0, \quad y'(x_0) = y'_0.$$

For each of the following values x_0, state the largest interval on which the corresponding IVP has a unique solution or state that no solution exists.

(a) $x_0 = \frac{1}{3}$

$(0, 2/3)$

(b) $x_0 = -\frac{1}{2}$

DNE

(c) $x_0 = \pi$

$(3, \infty)$

(d) $x_0 = \frac{\left(\frac{2}{3} + e\right)}{2}$

$(1, e)$

2. Consider the second-order homogeneous ODE $y'' - 2y' + y = 0$.

(a) (*5 pts*) Use Abel's theorem to compute the Wronskian of the solutions y_1 and y_2 of the ODE $y'' - 2y' + y = 0$. Do not solve the ODE!

SOLUTION:

$$y'' - 2y' + y = 0$$

$$p(x) = -2$$

Abel's: $W(y_1, y_2) = C \cdot e^{-\int p(x)\,dx}$

$$= C \cdot e^{-\int -2\,dx}$$

$$= C \cdot e^{+2x}$$

Part (b) is on the next page

(b) (*5 pts*) **True or False:** $y_1 = e^x$ and $y_2 = xe^x$ are solutions to the ODE $y'' - 2y' + y = 0$. Justify your claim!

SOLUTION:

$$y_1 = e^x \Rightarrow y_1'' - 2y_1' + y_1 = e^x - 2e^x + e^x = 0. \checkmark$$

$$y_2 = x \cdot e^x \Rightarrow y_2' = xe^x + e^x \Rightarrow y_2'' = xe^x + e^x + e^x$$
$$= xe^x + 2e^x$$

$$\Rightarrow y_2'' - 2y_2' + y_2 = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x$$
$$= 0. \checkmark$$

(c) (*5 pts*) **True or False:** $y_1 = e^x$ and $y_2 = xe^x$ form a *fundamental system of solutions* to the ODE $y'' - 2y' + y = 0$. Justify your claim!

SOLUTION: True !

By (b), y_1 & y_2 are both solutions, & by (a),

$$W(y_1, y_2) = Ce^{+2x} \neq 0 !$$

Part (d) is on the next page

(d) (*5 pts*) Using parts (a)—(c) above, use the method of variation of parameters to compute a particular solution $Y(t)$ of the non-homogeneous ODE $y'' - 2y' + y = \sin(x)$. **You must take any integrals you encounter, but otherwise, do not simplify!**

Hint:

- $\displaystyle\int xe^{-x}\sin x\,dx = -\frac{1}{2}xe^{-x}\sin x - \frac{1}{2}xe^{-x}\cos x - \frac{1}{2}e^{-x}\cos x$
- $\displaystyle\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}\sin x - \frac{1}{2}e^{-x}\cos x$

SOLUTION:

$$Y(t) = -y_1(t)\int \frac{y_2(t)\,g(t)}{W(y_1,y_2)}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W(y_1,y_2)}\,dt$$

$$= -e^t \int \frac{te^t \cdot \sin t}{ce^{2t}}\,dt + te^t \int \frac{e^t \sin t}{ce^{2t}}\,dt$$

$$= \frac{-e^t}{c}\int te^{-t}\sin t\,dt + \frac{te^t}{c}\int e^{-t}\sin t\,dt$$

$$= \frac{-e^t}{c}\left[\frac{-1}{2}te^{-t}\sin t - \frac{1}{2}te^{-t}\cos t - \frac{1}{2}e^{-t}\cos t\right] + \frac{te^t}{c}\left[\frac{-1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t\right]$$

3. (*15 pts*) Determine a suitable form (or "guess") for a particular solution $Y(t)$ of the non-homogeneous ODE

$$y'' - 3y' + 4y = \underbrace{(t^2 - 1)\sin t}_{①} + \underbrace{t^3 \cos 3t}_{②} - \underbrace{e^{-t}\sin 2t}_{③}$$

if the method of undetermined coefficients is to be used. Do not solve for the coefficients!

SOLUTION:

① $(At^2 + Bt + C)\sin(t) + (Dt^2 + Et + F)\cos t$

② $(At^3 + Bt^2 + Ct + D)\cos(3t) + (Et^3 + Ft^2 + Gt + H)\sin(3t)$

③ $Ae^{-t}\sin(2t) + Be^{-t}\cos(2t)$.

4. (*10 pts ea.*) If it exists, find the Laplace transform $F(s)$ corresponding to each of the following functions $f(t)$ and simplify fully! Otherwise, indicate that no Laplace transform exists and explain why.

Only do three (3) of the four parts, and clearly state which three you're doing!

(a) $f(t) = \begin{cases} e^{-2t} & \text{if } 0 <= t < 4 \\ 7 & \text{if } t = 4 \\ 0 & \text{if } t > 4 \end{cases}$

SOLUTION:

$$F(s) = \int_0^4 e^{-st} e^{-2t}\,dt + \int_4^4 e^{-st}(7)\,dt + \int_4^\infty e^{-st}(0)\,dt$$

$$= \int_0^4 e^{-t(2+s)}\,dt$$

$$= \frac{1}{-(2+s)} e^{-t(2+s)}\Big]_{t=0}^{t=4}$$

$$= \frac{1}{-2-s}\left[e^{-4(2+s)} - 1\right]$$

$$= \frac{1}{-2-s}\left[e^{-8-4s} - 1\right]$$

Part (b) is on the next page

∞ instead of 4: -2

(b) $f(t) = t^t$

SOLUTION:

Does not exist!

Part (c) is on the next page

(c) $f(t) = \cos t$

Hint: $\cos t = \dfrac{e^{it} + e^{-it}}{2}$ (where $i = \sqrt{-1}$ is the imaginary unit), though you don't have to use this.

SOLUTION:

Old way

$u = e^{-st}$ $\quad v = \sin t$

$u' = -se^{-st}$ $\quad v' = \cos t$

$$\int_0^\infty e^{-st} \cos t \, dt$$

$$= e^{-st} \sin t \Big]_0^\infty + s \int_0^\infty e^{-st} \sin t \, dt$$

$u = e^{-st}$ $\quad v = -\cos t$

$u' = -se^{-st}$ $\quad v' = \sin t$

$$= e^{-st} \sin t \Big]_0^\infty + s \left(-e^{-st} \cos t \Big]_0^\infty - s \int_0^\infty e^{-st} \cos t \, dt \right)$$

$$\Rightarrow F(s) = \frac{e^{-st} \sin t \Big]_0^\infty - s e^{-st} \cos t \Big]_0^\infty}{1 + s^2}$$

$$\Rightarrow F(s) = \frac{s}{1 + s^2}$$

New way

$$\int_0^\infty e^{-st} \left(\frac{e^{it} + e^{-it}}{2} \right) dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t(s-i)} \, dt + \frac{1}{2} \int_0^\infty e^{-t(s+i)}$$

$$= \frac{1}{2} \left(\frac{-1}{s-i} e^{-t(s-i)} \Big]_0^\infty + \frac{-1}{s+i} e^{-t(s+i)} \Big]_0^\infty \right)$$

$$= \frac{1}{2} \left(\frac{+1}{s-i} + \frac{1}{s+i} \right)$$

$$= \frac{1}{2} \left(\frac{s+i + s-i}{(s+i)(s-i)} \right)$$

$$= \frac{1}{2} \left(\frac{2s}{s^2 + \cancel{si} - \cancel{si} - i^2} \right)$$

$$= \frac{1}{2} \, \frac{2s}{s^2 + 1}$$

$$= \frac{s}{1 + s^2}$$

Part (d) is on the next page

(d) $f(t) = t$

SOLUTION:

$$\int_0^\infty t e^{-st}\, dt = -\frac{1}{s} t e^{-st}\Big]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} t e^{-st}\Big]_0^\infty - \frac{1}{s^2} e^{-st}\Big]_0^\infty$$

$$= \frac{1}{s^2}$$

5. (*3 pts ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required, and no credit will be given if you write *only* the letters "T" or "F"!**

(a) The IVP

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_0'$$

always has a unique solution.

False.

(b) A Laplace transform $F(s)$ exists for every function $f(t)$.

False.

(c) If y_1 and y_2 are solutions of the second-order homogeneous linear ODE $y'' + p(x)y' + q(x)y = 0$ corresponding to the non-homogeneous ODE $y'' + p(x)y' + q(x)y = g(t)$, then the function

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y_1, y_2)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y_1, y_2)}\,dt$$

satisfies the relationship $Y''(t) + p(t)Y'(t) + q(t)Y(t) = g(t)$.

False.

(d) Laplace transforms are unique. In other words: If a function $f(t)$ has Laplace transform $F(s)$, then no other function $g(t)$ can have $F(s)$ as its Laplace transform.

False

(e) If e^{at} and te^{at} are solutions of the homogeneous ODE $y'' + p(t)y' + q(t)y = 0$, then the suitable form for a particular solution $Y(t)$ of the ODE $y'' + p(t)y' + q(t)y = 4e^{at}$ according to the method of undetermined coefficients is $Y(t) = (At^2 + Bt + C)e^{at}$.

False

Bonus: The *gamma function* of any real number p is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1}\, dt;$$

this integral converges for every value t in $(0, \infty)$ when $p > -1$. Consider the following four (4) bonus questions about the gamma function. (*5 pts ea.*)

(a) Use integration by parts to show that $\Gamma(p+1) = p\Gamma(p)$.

Hint: Do not write "$\Gamma(p+1) = p \cdot \Gamma(p)$" as your first step; "$0 = 0$" is not a valid answer!

SOLUTION:

Part (b) is on the next page

(b) Using the result from part (a), show that the Laplace transform of $f(t) = t^{p-1}$ is equal to $\dfrac{\Gamma(p)}{s^p}$.

Hint: Do a u-substitution with $u = st$.

SOLUTION:

Part (c) is on the next page

(c) Show that the Laplace transform of $f(t) = t^{-1/2}$ is equal to $\displaystyle\int_0^\infty e^{-v^2}\,dv$.

Hint: First, do a u-substitution with $u = st$; afterwards, do a v-substitution (the same as u-sub but with a different letter) of the form $v = u^{1/2}$.

SOLUTION:

Part (d) is on the next page

(d) Use the result of part (c) along with the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ to show that the Laplace transform of $f(t) = t^{-1/2}$ is $F(s) = \sqrt{\frac{\pi}{s}}$.

SOLUTION:

Scratch Paper