Exam 2

Jun 26, 2017

MAP 2302—ODE, Summer 2017

(Clearly!) Print Name: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **7 problems** and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by circling them and/or drawing a box around them.
   - More than one choice may be a correct answer for a question; if so, circle all correct answers!
   - There may be correct answers which aren't listed; in this case, only focus on the choices provided!
8. Some questions are True/False.
   - If you write *True*, you should give a "proof" or (thorough!) explanation of why.

     Example: "All quadratic functions have derivatives which are linear" is *True*, and the proof is: If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$, which is linear.
   - If you write *False*, you should give and explain a counterexample.

     Example: "All polynomials have graphs which are parabolas" is *False*; a counterexample is the function $f(x) = x^3$, whose graph *isn't* a parabola, and I could "explain" why this is a counterexample by drawing the non-parabola graph of $y = f(x)$.
9. The notation $y^{(n)}$ **always** means "the n^{th} derivative of y", e.g. $y^{(4)}$ is the fourth derivative of y and is equivalent to y'''' (with 4 primes).

Question	1 (15)	2 (10)	3 (10)	4 (10)	5 (15)	6 (15)	7 (25)	Total (100)
Points								

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*15 pts*) Find the general solution of the ODE

$$\left(e^y \sin y + \cos x - 2x^2y + y^3\right) y' = 2xy^2 + y\sin x + x^2e^x + 4$$

Hint: $\int t^2e^t\,dt = (t^2 - 2t + 2)e^t + C.$

$M = -2xy^2 - y\sin x - x^2e^x - 4$
$N = e^y \sin y + \cos x - 2x^2y + y^3$

SOLUTION: • Is it exact?

$M_y = -4xy - \sin x \qquad N_x = -\sin x - 4xy$

• Know: There exists $f(x,y)$ s.t.

- $f_x = M = -2xy^2 - y\sin x - x^2e^x - 4$ (*)
- $f_y = N = e^y \sin y + \cos x - 2x^2y + y^3$ (**)
- Gen soln: $f(x,y) = C$.

• Find f:

- Using (*): $f = -x^2y^2 + y\cos x - (x^2-2x+2)e^x - 4x + h(y)$ (***)
- Using (***): $f_y = -2x^2y + \cos x + h'(y)$
- Compare w/ (**): $-2x^2y + \cos x + h'(y) = e^y\sin y + \cos x - 2x^2y + y^3$

$\Rightarrow h'(y) = e^y \sin y + y^3$

$\Rightarrow h(y) = \int e^y \sin y\,dy + \frac{1}{4}y^4$

Gen soln:

$$-x^2y^2 + y\cos x - (x^2-2x+2)e^x - 4x + \int e^y \sin y\,dy + \frac{1}{4}y^4 = C.$$

$= \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x$

Exact: 4
f_x &/or f_y: 2
f w/ h : 2 ⇝ der w/ h' : 2
Int h'=h : 2
Ans: 3 ($\int e^y \sin y\,dy = +5$)

2. (*2 pts ea.*) Consider the first-order IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

Indicate whether each of the following questions is True or False by writing the words "True" or "False".
No justification is required, and no credit will be given if you write *only* the letters "T" or "F"!

(a) The IVP may fail to have a solution if f is continuous in a rectangle containing x_0.

False.

(b) The IVP always has a *unique* solution if f and $\dfrac{\partial f}{\partial x}$ are both continuous in a rectangle containing x_0.

False.

(c) Under no circumstance may the IVP have multiple solutions.

False.

(d) If f and $\dfrac{\partial f}{\partial y}$ are both continuous in a rectangle containing x_0, then two integral curves of the ODE $\dfrac{dy}{dx} = f(x,y)$ in the xy-plane may both pass through the point (x_0, y_0).

False.

(e) If $f(x,y) = -7$, then the IVP has a unique solution.

True.

3. (*10 pts*) The ODE

$$\left(\frac{x}{y} - \sin y\right) y' + 3 = 0$$

is not exact. Find an integrating factor which makes it exact. **Simplify fully but do not solve the ODE!**

SOLUTION: $M = 3 \quad N = \frac{x}{y} - \sin y$

Note: $\frac{N_x - M_y}{M} = \frac{\frac{1}{y} - 0}{3} = \frac{1}{3y}$ contains only y's,

So $\exp\left(\int \frac{1}{3y}\,dy\right) = \exp\left(\frac{1}{3}\int \frac{1}{y}\,dy\right)$

$= \exp\left(\frac{1}{3}\ln|y|\right)$

$= |y|^{1/3}$ is an integrating factor

$= C y^{1/3}$ for $C = \pm 1$.

Check:

$y^{1/3}\left(\frac{x}{y} - \sin y\right)y' + 3y^{1/3} = 0 \Rightarrow \underbrace{\begin{array}{l} M = 3y^{1/3} \Rightarrow M_y = y^{-2/3} \\ N = \frac{x}{y^{2/3}} - y^{1/3}\sin y \Rightarrow N_x = y^{-2/3}. \end{array}}_{\text{is exact!}}$

Ratio: 5
m(x): 5
Check: +2

$$\frac{dy}{dx} = \underbrace{\frac{\ln(\ln(2x))+2\ln x}{y\cos y}}_{f(x,y)}$$

4. (*10 pts*) For which of the following initial conditions does the IVP

$$(y\cos y)\frac{dy}{dx} - 2\ln x = \ln(\ln(2x)), \quad y(x_0) = y_0$$

have a unique solution? **There may be more than one!**

i. $y\left(\frac{1}{2}\right) = 2$

ii. $y\left(\frac{1}{4}\right) = 2$

iii. $y\left(\frac{1}{4}\right) = \frac{\pi}{2}$

iv. $y\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$

v. $y\left(\frac{e}{2}\right) = 0$

vi. $y\left(\frac{e}{2}\right) = \frac{3\pi}{2}$

vii. $y\left(\frac{\pi}{2}\right) = \frac{3e}{2}$

viii. $y\left(-\frac{e}{2}\right) = \frac{\pi}{4}$

ix. None of These

ok: $x_0 > \frac{1}{2}$

$y_0 \neq 0$

$\neq \frac{n\pi}{2}$

f Not continuous: • $y=0$ • $\cos y = 0 \Rightarrow y = \frac{n\pi}{2}$, n integer

• $x \leq 0$ • $\ln 2x \leq 0 \Rightarrow 2x \leq 1 \Rightarrow x \leq \frac{1}{2}$.

$$\frac{\partial f}{\partial y} = \frac{0-(\ln(\ln(2x))+2\ln x)(-y\sin y+\cos y)}{y\cos y}$$

discontinuous @ same vals!

5. (*5 pts ea.*) For each of the following functions y, determine a second-order linear homoogeneous ODE having y as a general solution.

Hint: $(x-(a+bi))(x-(a-bi)) = (x-a)^2 + b^2$ for all real numbers a and b.

(a) $y = c_1 e^{-2t} + c_2 t e^{-2t}$ → char eq. w/ roots $r_1 = r_2 = -2$

↳ $(r+2)(r+2)=0 \Rightarrow r^2+4r+4=0$

$$\boxed{\text{ODE: } y'' + 4y' + 4y = 0.}$$

(b) $y = e^t(c_1 \cos t + c_2 \sin t)$ Char eq. w/ roots $1 \pm i$.

↳ $(r-(1+i))(r-(1-i)) = 0$

$\Downarrow$

$0 = (r-1)^2 + 1^2 = r^2 - 2r + 1 + 1$

$= r^2 - 2r + 2.$

$$\boxed{\text{ODE: } y'' - 2y' + 2y = 0.}$$

(c) $y = c_1 e^{5t} + c_2 e^{-5t}$ → Char eq. w/ roots $r_1 = 5$ & $r_2 = -5$

↳ $(r-5)(r+5) = 0 \Rightarrow r^2 - 25 = 0$

$$\boxed{\text{ODE: } y'' - 25y = 0}$$

6. (*15 pts*) Solve the IVP

$$2y'' - 3y' - 5y = 0, \quad y(0) = 1, \quad y'(0) = 4.$$

SOLUTION:

5

- Char eq: $2r^2 - 3r - 5 = 0 \Rightarrow 2r^2 + 2r - 5r - 5 = 0$

$\Rightarrow 2r(r+1) - 5(r+1) = 0$

$\Rightarrow (2r-5)(r+1) = 0$

$\Rightarrow r = \frac{5}{2} \quad r = -1$

5

- Gen soln: $y = c_1 e^{5/2x} + c_2 e^{-x}$ $\left(\Rightarrow y' = \frac{5}{2} c_1 e^{5/2x} - c_2 e^{-x}\right)$

5

- IVP: $1 = c_1 + c_2$

$4 = \frac{5}{2} c_1 - c_2$

$\Rightarrow 5 = \frac{7}{2} c_1 \Rightarrow c_1 = \frac{10}{7}$

$c_2 = \frac{-3}{7}$

$$\Rightarrow \boxed{y = \frac{10}{7} e^{5/2x} - \frac{3}{7} e^{-x}}$$

7. Answer the following four questions about the pair of functions $y_1 = \cos 2x$ and $y_2 = \sin 2x$.

(a) (*5 pts*) Verify that y_1 and y_2 are solutions of the ODE $y'' + 4y = 0$.

$y_1 = \cos 2x$

$\Rightarrow y_1' = -2\sin 2x$

$\Rightarrow y_1'' = -4\cos 2x$

So: $y_1'' + 4y_1 = -4\cos 2x + 4\cos 2x$

$= 0.$

$y_2 = \sin 2x$

$\Rightarrow y_2' = 2\cos 2x$

$\Rightarrow y_2'' = -4\sin 2x$

So: $y_2'' + 4y_2 = -4\sin 2x + 4\sin 2x$

$= 0.$

(b) (*10 pts*) Find the Wronskian of y_1 and y_2.

$$W(y_1, y_2) = \det\begin{pmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{pmatrix}$$

$$= 2\cos^2 2x - (-2\sin^2 2x)$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2(1)$$

$$= 2.$$

Part (c) is on the next page

(c) (*5 pts*) Do the functions y_1 and y_2 constitute a fundamental set of solutions for the ODE $y'' + 4y = 0$? Why or why not?

Yes!

y_1 & y_2 are both solutions, by (a) and $W(y_1, y_2) = 2 \neq 0$! by (b)

Thus, by def, $\{y_1, y_2\}$ is a fund. sys. of solutions.

(d) (*5 pts*) **True or False:** If y_3 is any solution to the ODE $y'' + 4y = 0$, then there exist constants c_1 and c_2 such that $y_3 = c_1 y_1 + c_2 y_2$. **Justify your claim!**

True.

By the theorem from class, $\{y_1, y_2\}$ being a fund. sys. of solutions means every solution has the form $c_1 y_1 + c_2 y_2$.

Scratch Paper