# Exam 1

Jun 5, 2017

MAP 2302—ODE, Summer 2017

(Clearly!) Print Name: ~ KEY ~

## Read <u>all</u> of what follows <u>carefully</u> before starting!

1. This test has **6 problems** and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook <u>as long as you fully and correctly state the result and where it came from.</u>
   - If you use a result/theorem, you have to state *which* result you're using <u>and</u> explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by <u>circling them</u> and/or <u>drawing a box around them.</u>
   - More than one choice may be a correct answer for a question; if so, circle <u>all</u> correct answers!
   - There may be correct answers which aren't listed; in this case, <u>only</u> focus on the choices provided!
8. Some questions are True/False.
   - If you write *True*, you should <u>give a "proof" or (thorough!) explanation of why.</u>

     **Example:** "All quadratic functions have derivatives which are linear" is *True*, and the proof is: If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$, which is linear.
   - If you write *False*, you should <u>give and explain a counterexample.</u>

     **Example:** "All polynomials have graphs which are parabolas" is *False*; a counterexample is the function $f(x) = x^3$, whose graph *isn't* a parabola, and I could "explain" why this is a counterexample by drawing the non-parabola graph of $y = f(x)$.
9. The notation $y^{(n)}$ **always** means "the $n^{\text{th}}$ derivative of $y$", e.g. $y^{(4)}$ is the fourth derivative of $y$ and is equivalent to $y''''$ (with 4 primes).

| Question | 1 (25) | 2 (5) | 3 (10) | 4 (20) | 5 (25) | 6 (15) | Total (100) |
|---|---|---|---|---|---|---|---|
| Points | | | | | | | |

**<u>Do not</u> write in these boxes! If you do, you get 0 points for those questions!**

1. Consider the autonomous ODE

$$\frac{dy}{dx} = y^3 - 9y. \quad = y(y^2-9)$$

(a) (*5 pts*) What are the equilibrium solutions of this ODE?

SOLUTION:

$y = -3$

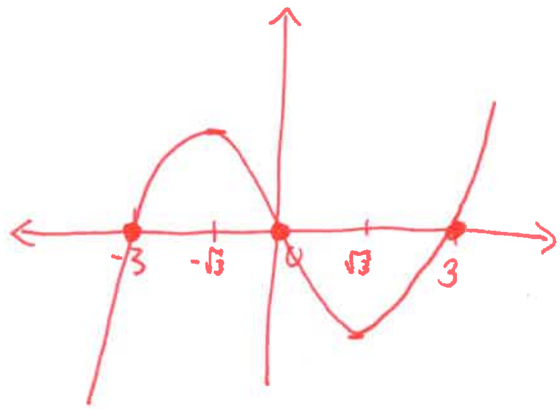
$y = 0$

$y = 3$

Note: $f'(y) = 3y^2 - 9 = 0$

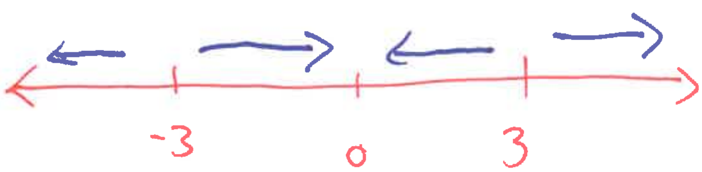
$\Leftrightarrow y^2 = 3$

$\Leftrightarrow y = \pm\sqrt{3}$




(b) (*5 pts*) Classify each of the equilibrium solutions of this ODE as asymptotically stable, asymptotically unstable, or neither. **Hint**: Do not assume that $y > 0$!

SOLUTION:



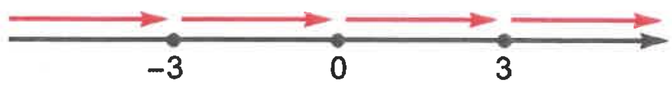

-3: unstable

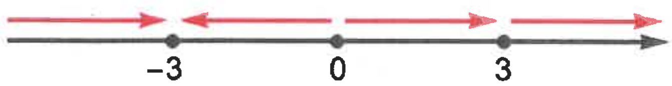
0: stable

3: unstable

**Part (c) is on the next page**

(c) (*5 pts*) Which of the following phase lines correspond to this ODE?

i) 


ii) 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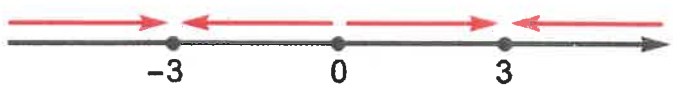
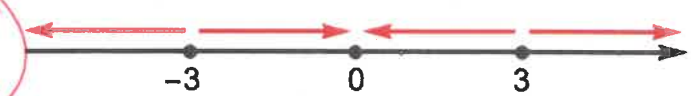
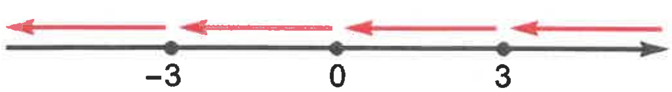
iv) 


v) 


vi) 


vii) None of the above

**Part (d) is on the next page**

(d) (*5 pts*) Assuming that $y(0) = 2$, find $\lim_{x \to \infty} y(x)$. **Do not solve the ODE!**

SOLUTION:

$$\lim_{x \to \infty} y(x) = 0.$$

(e) (*5 pts*) **True or False:** This ODE is separable. **Justify your claim!**

**Hint**: See item 8 on the front page for what constitutes *justification*!

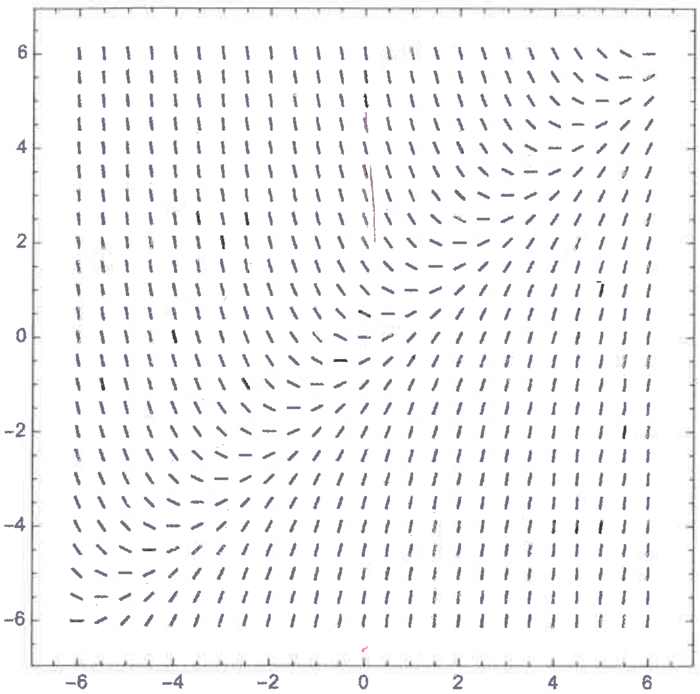
SOLUTION:

True.

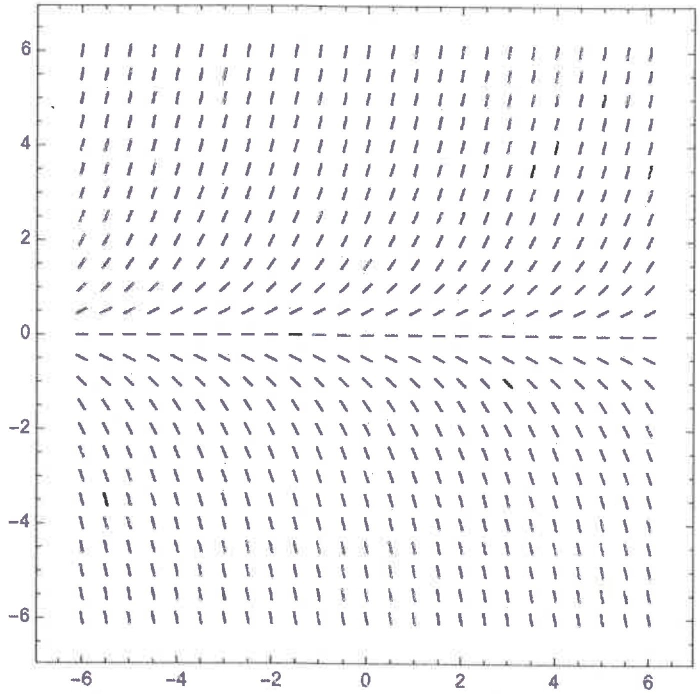
- Every autonomous ODE is separable
- separate it:

$$\frac{dy}{y^3 - 9y} = dx$$

2. (*5 pts*) Which of the following slope fields correspond to the ODE $\dfrac{dy}{dx} = x + y$?




(i)

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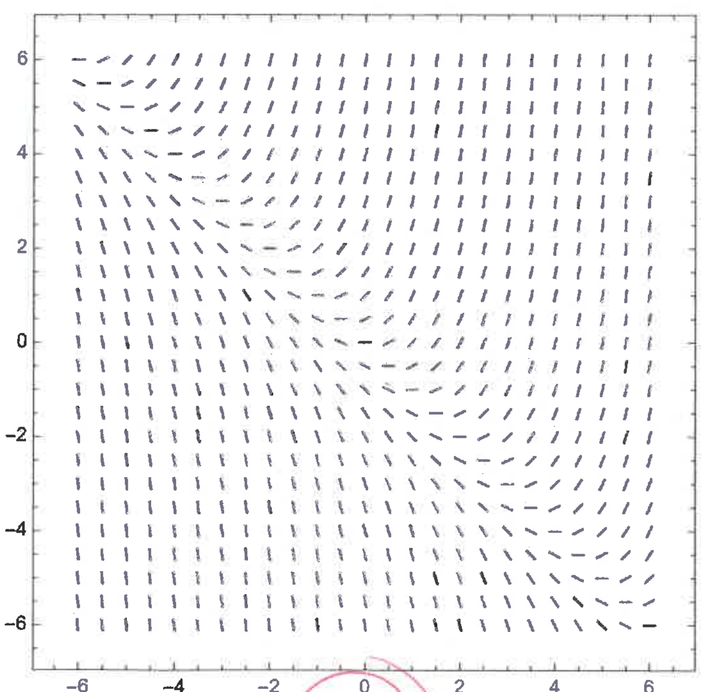
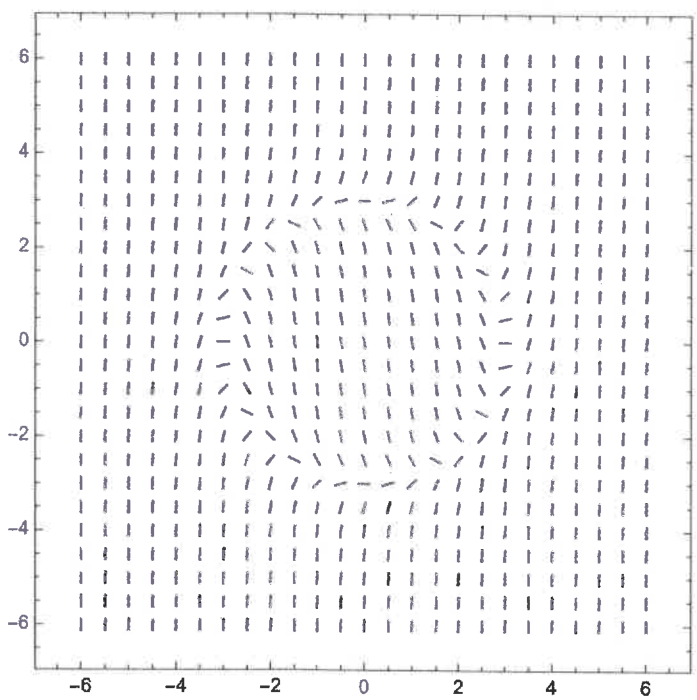



(iii)




(iv)

3. (*10 pts*) Show that $y = \sin x + \cos x$ is a solution to the third-order ODE

$$y^{(3)} - y'' + y' = y.$$

SOLUTION:

$y = \sin x + \cos x$

$y' = \cos x - \sin x$

$y'' = -\sin x - \cos x$

$y^{(3)} = -\cos x + \sin x$

$y''' - y'' + y' =$

$\sin x - \cos x$

$-(-\sin x - \cos x) = \underbrace{\sin x + \cos x}_{= y!}$

$+ \cos x - \sin x$

der's = 2pts ea

Note: Don't write "=" until you know they are!

4. (*10 pts ea.*) Answer the following questions about the first-order linear ODE

$$(x - x^3)y' + y - x^3 = 0. \Rightarrow y' + \frac{1}{x-x^3}y = \frac{x^3}{x-x^3}$$

(a) Find the integrating factor of the differential equation. **Simplify fully!**

**Hint**: Expect to use partial fractions; also, be careful with your algebra: Things should be nice!

SOLUTION: $m(x) = \exp\left(\int p(x)dx\right) = \exp\left(\int \frac{1}{x-x^3}dx\right) = \exp\left(\int \frac{1}{x(1-x)(1+x)}dx\right)$.

- Using partial fractions: $\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \Rightarrow 1 = A(1+x)(1-x) + Bx(1+x) + Cx(1-x)$.

$\hookrightarrow$ @ $x=0$: $1 = A+0+0 \Rightarrow A=1$

@ $x=1$: $1 = 0+2B+0 \Rightarrow B=\frac{1}{2}$

@ $x=-1$: $1 = 0+0-2C \Rightarrow C=\frac{-1}{2}$

So: $\int \frac{1}{x(1-x)(1+x)}dx = \int \frac{1}{x} + \frac{\frac{1}{2}}{1-x} - \frac{1/2}{1+x}\,dx$

$= \ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|$

$= \ln\left(\frac{|x|}{\sqrt{|1-x|}\sqrt{|1+x|}}\right)$

- plug in to m: $m(x) = \exp\left(\ln\left(\frac{|x|}{\sqrt{|1-x|}\sqrt{|1+x|}}\right)\right)$

$= \frac{|x|}{\sqrt{|1-x| \cdot |1+x|}}$

(b) Determine which of the following is the largest interval on which the IVP $(x - x^3)y' + y - x^3 = 0$, $y(\underbrace{-0.5}_{x_0}) = 3$ exists and is unique. **Do not solve the ODE!**

i. $(-\infty, -1)$ ii. $(-1, 1)$ iii. $(-1, 0)$ iv. $(0, 1)$ v. $(1, \infty)$ vi. $(-\infty, 0)$

$p(x) = \frac{1}{x(1-x)(1+x)}$ $\rightsquigarrow$ continuous for all $\mathbb{R}$ except $0, -1, 1$

$q(x) = \frac{x^3}{x(1-x)(1+x)}$ $\rightsquigarrow$ continuous for all $\mathbb{R}$ except $0, -1, 1$

$\hookrightarrow$ both continuous on $(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$

$\uparrow$

$x_0 = -0.5$ lives here

$\Rightarrow$ $(-1, 0)$ is the max interval

5. (*5 pts ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False" (and **not** just the letters "T" or "F"). **Justify your answers!**

**Hint**: See item 8 on the front page for what constitutes *justification*!

(a) Every separable ODE is exact.

• what I was going for: True, b/c $f(x)dx = g(y)dy \Rightarrow \underbrace{g(y)}_{N}\frac{dy}{dx} \underbrace{- f(x)}_{M} = 0 \Rightarrow \left.\begin{matrix} M_y = 0 \\ N_x = 0 \end{matrix}\right] \Rightarrow$ exact

• As written: False. If ODE autonomous, $\frac{dy}{dx} = f(y) \Rightarrow \frac{dy}{dx} - f(y) = 0 \Rightarrow \left.\begin{matrix} M = -f(y) \\ N = 1 \end{matrix}\right] \Rightarrow \begin{matrix} M_y = -f'(y) \\ N_x = 0 \end{matrix}$, so this isn't continuous. (The curve gives everyone credit here)

(b) The (first-order) linear ODE $\frac{dy}{dx} + p(x)y = q(x)$ has a unique (i.e. exactly one) integrating factor.

False: ~~The~~ Any constant multiple of $\exp(\int p(x)\,dx)$ is an integrating factor. (To see this: add the "+C" to the integral ....)

(c) Every separable ODE is autonomous.

False: $\frac{dy}{dx} = x$ is separable ($\Rightarrow dy = x\,dx$) but not autonomous.

(d) Every (first-order) linear ODE is exact.

False: You can almost pick a counter-example at random, e.g.

$\frac{dy}{dx} + y = 4 \Rightarrow \frac{dy}{dx} + y - 4 = 0 \Rightarrow \left.\begin{matrix} M = y-4 \\ N = 1 \end{matrix}\right] \Rightarrow \left.\begin{matrix} M_y = 1 \\ N_x = 0 \end{matrix}\right] \Rightarrow$ not exact.

Note: This is linear w/ $p=1$ & $q=4$.

(e) Let $r > 0$ be a constant. The general solution of the ODE $\frac{dy}{dx} = -ry$ is $y = e^{-rx} + C$, where $C$ is a constant.

False. $\frac{dy}{dx} = -ry \Rightarrow \frac{dy}{y} = -r\,dx \Rightarrow \int \frac{dy}{y} = \int -r\,dx \Rightarrow \ln|y| = -rx + C$

$\Rightarrow |y| = e^{-rx+C} = e^{-rx}e^{C} = Ce^{-rx}$ (where "C" = $e^C$ from before)

$\Rightarrow y = Ce^{-rx}$. This is not the same as $e^{-rx} + C$!

6. (*15 pts*) Solve the IVP

$$\underbrace{(x\cos y + 3y^2)}_{N}y' + \underbrace{\sin y + 2x}_{M} = 0, \quad y(0) = \pi^3.$$

SOLUTION:

$M_y = \cos y \qquad N_x = \cos y \implies$ exact!

• Know: There is a function $f(x,y)$ such that

$f_x = M \implies f_x = \sin y + 2x \quad (\star)$

$f_y = N \implies f_y = x\cos y + 3y^2 \quad (\star\star).$

• Find f: (i) Integrate $(\star)$ WRT $x$:

$f = x\sin y + x^2 + h(y)$, some $h$ $(\star\star\star)$

(ii) Find $f_y$: from $(\star\star\star)$

$f_y = x\cos y + h'(y)$

(iii) Equate w/ $(\star\star)$:

$x\cos y + h'(y) = x\cos y + 3y^2 \implies h'(y) = 3y^2$

$\implies h(y) = y^3$

$\implies f(x,y) = x\sin y + x^2 + y^3.$

• Solve ODE: general Solution is $f = C \implies \boxed{x\sin y + x^2 + y^3 = C}$

• Find C: $(0,\pi^3) \implies 0 + 0 + (\pi^3)^3 = C \implies C = \pi^9$

$\implies$ particular solution: $\boxed{x\sin y + x^2 + y^3 = \pi^9}.$

Exact = 3
f w/ h(y) = 2
$f_y$ compare = 2
f = 2 → gen = 3
IVP = 3 (C = 2)

**Bonus:** If the right side of a first-order ODE can be written as a function of the ratio $y/x$ only, then the equation is said to be *homogeneous*. For example,

$$\frac{dy}{dx} = \frac{y-4x}{x-y} = \frac{(y/x)-4}{1-(y/x)} \quad \text{(by dividing everything by } x\text{)},$$

and so this ODE is homogeneous. To solve such an ODE, the steps are as follows:

(i) Introduce the function $v(x) = y/x \iff y = xv(x)$;

(ii) calculate $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$; and

(iii) replace all instances of $y$ and $\dfrac{dy}{dx}$ in the original ODE with these expressions of $x$, $v$, and $\dfrac{dv}{dx}$.

The resulting ODE will be separable.

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Use the above information to answer the following questions about the ODE $\dfrac{dy}{dx} = \dfrac{x^2+3y^2}{xy}$.

(a) (*1 pt*) Show that this ODE is homogeneous. **Hint**: Divide everything by $x^2$.

(b) (*1 pt*) Let $v(x) = y/x \iff y = xv(x)$. Show that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

**Part (c) is on the next page**

(c) (*1 pt*) Use parts (a) and (b) to rewrite the original ODE in terms of $x$, $v$, and $\dfrac{dv}{dx}$.

(d) (*1 pt*) Show that the ODE in part (c) is separable.

**Part (e) is on the next page**

(e) (*5 pts*) Solve the ODE from part (c) in terms of $x$, $v$, and $\frac{dv}{dx}$.

(f) (*1 pt*) Find the solution of the original ODE by replacing $v$ by $y/x$ in (e).

Scratch Paper

$$f(y)\,dy = g(x)\,dx$$

$$\Rightarrow \quad f(y)\frac{dy}{dx} = g(x)$$

$$\Rightarrow \quad \underbrace{f(y)}_{N}\frac{dy}{dx} - \underbrace{g(x)}_{M} = 0$$

$$M_y = 0$$

$$N_x = 0$$

$$\underbrace{\frac{dy}{dx}}_{N} - \underbrace{\frac{g(x)}{f(y)}}_{M} = 0$$

$$\frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$$

$$\rightsquigarrow 1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$$

@0: $A = 1$

@1: $1 = 2B \to B = \frac{1}{2}$

@-1: $1 = -2C \to C = \frac{-1}{2}$

$$\int \frac{1}{x} + \frac{1/2}{1-x} + \frac{-1/2}{1+x}$$

$$\ln|x| - \frac{1}{2}\ln|1-x| - \frac{1}{2}\ln|1+x|$$

$$\frac{x}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$x = A(1+x) + B(1-x)$$

$$\frac{1}{x} + \frac{x}{1-x^2}$$

@ 1: $1 = 2A$ $\quad A = \frac{1}{2}$

@ −1: $-1 = 2B$ $\quad B = \frac{-1}{2}$

$$\parallel$$

$$\frac{1-x^2+x^2}{x(1-x^2)} = \frac{1}{x-x^3}$$

$$\frac{1}{x} + \frac{x}{1-x^2}$$

$\underline{\text{④}}$

$$\frac{1}{x(1-x^2)} = \frac{A}{x} + \frac{Bx+C}{1-x^2}$$

$$\Rightarrow 1 = A(1-x^2) + x(Bx+c)$$

@ 0: $A = 1$

@ 1: $1 = B+C$

@ −1: $1 = -1(C-B)$

$1 = B+C$

$1 = B-C$

$0 = 2C$

$\Rightarrow C = 0$

$B = 1$