**Quiz 2**/**test prep 1**
(front and back)

Name: ~ KEY ~
(please print neatly!)

**Directions:** Answer each of the following questions. Make sure to read the instructions for each question as you proceed. *For multiple choice questions, indicate your choice(s) by circling/drawing a box around the appropriate selection(s).*

Throughout, consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} -x_2 \\ 0 \\ x_1 \\ x_1 + x_3 \end{pmatrix}$.

1. **True or False:** $T$ is a linear transformation. Justify your claim.

Check: $\underbrace{T(c\vec{u}+d\vec{v})}_{LHS} = \underbrace{cT(\vec{u})+dT(\vec{v})}_{RHS}$

True. Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$LHS = T\begin{pmatrix} cu_1+dv_1 \\ cu_2+dv_2 \\ cu_3+dv_3 \end{pmatrix} = \begin{pmatrix} -(cu_2+dv_2) \\ 0 \\ cu_1+dv_1 \\ (cu_1+dv_1)+(cu_3+dv_3) \end{pmatrix} = \begin{pmatrix} -cu_2 \\ 0 \\ cu_1 \\ cu_1+cu_3 \end{pmatrix} + \begin{pmatrix} -dv_2 \\ 0 \\ dv_1 \\ dv_1+dv_3 \end{pmatrix}$$

$$= c\begin{pmatrix} -u_2 \\ 0 \\ u_1 \\ u_1+u_3 \end{pmatrix} + d\begin{pmatrix} -v_2 \\ 0 \\ v_1 \\ v_1+v_3 \end{pmatrix} = cT(\vec{u}) + dT(\vec{v}) = RHS.$$

2. Compute:

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

3. Find the canonical matrix A corresponding to the transformation $T$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ or state that no such matrix exists.

Using 2: $A = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

4. What is the domain of $T$?

$\mathbb{R}^3$

5. What is the codomain of $T$?

$\mathbb{R}^4$

E.C.

6. Find/describe the range of $T$.

**Hint**: You can look at the right-hand side of $T$ and write a *parametric vector form* for $T$; this will suffice!

- Range $(T)$ = subset of $\mathbb{R}^4$ which looks like $r\vec{v}_1 + s\vec{v}_2 + t\vec{v}_3$ $(r, s, t \in \mathbb{R})$ for $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ = cols$(A)$.
- Because $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ L.I., the range$(T)$ is a copy of $\mathbb{R}^3$ in $\mathbb{R}^4$ (b/c span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an $\mathbb{R}^3$).

7. Is the codomain of $T$ equal to the range of $T$? How do you know? If they *aren't* the same, find a point in codomain($T$) that *isn't* in range($T$). No, they're not.

Ex: $\begin{pmatrix} \square \\ 1 \\ \square \\ \square \end{pmatrix}$ isn't in range$(T)$.

$$\begin{pmatrix} -x_2 \\ 0 \\ x_1 \\ x_1+x_3 \end{pmatrix} = x_1 \overbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}^{\vec{v}_1} + x_2 \overbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\vec{v}_2} + x_3 \overbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}^{\vec{v}_3} = r\vec{v}_1 + s\vec{v}_2 + t\vec{v}_3 \quad (r, s, t \in \mathbb{R})$$

8. Is $T$ injective/one-to-one? Justify your claim.

Is one-to-one: • By observation, cols $(A) = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are L.I. or

• Consider $A\vec{x} = \vec{0} \iff \left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\substack{R_3 \leftrightarrow R_1 \\ R_1 \leftrightarrow R_2 \\ R_4 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_4}]{} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{R_3 = R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$; so $A\vec{x} = \vec{0}$ has only the trivial solution.

9. Is $T$ surjective/onto? Justify your claim.

Not onto: • By (7), range $(T) \neq$ codomain $(T)$. or

• For example: No $x \in \mathbb{R}^3$ s.t. $T(\vec{x}) = \begin{pmatrix} \square \\ 1 \\ \square \\ \square \end{pmatrix} \ldots$

Scratch Paper