Ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \rightsquigarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{pmatrix} \rightsquigarrow (1-\lambda)^2 - 2$ [char poly]

So eigenvalues are: $(1-\lambda)^2 - 2 = 0 \Rightarrow \lambda^2 - 2\lambda + 1 - 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 1 = 0,$

i.e. $\lambda = \dfrac{2 \pm \sqrt{4 - 4(-1)}}{2} = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$

Note: $\begin{pmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1-\lambda \\ 1-\lambda & 2 \end{pmatrix} \rightsquigarrow \underbrace{\begin{pmatrix} 1 & 1-\lambda \\ 0 & 1+2\lambda-\lambda^2 \end{pmatrix}}_{\text{matrix} = A}$

- $\lambda = 1 + \sqrt{2}$:

  $x_1 = -(1-\lambda)x_2$
  $x_2 = x_2$
  $\Rightarrow \vec{x} = x_2 \begin{pmatrix} -1 + (1+\sqrt{2}) \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} x_2$

  let $E_1 = \text{span}\{<\sqrt{2}, 1>\}$ (= this eigenspace)

  Basis for eigenspace: $\{<\sqrt{2}, 1>\}$

- $\lambda = 1 - \sqrt{2}$: Similarly,

  $\vec{x} = x_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$ & $\underbrace{\text{basis} = \{<-\sqrt{2}, 1>\}}_{\text{let } E_2 = \text{span}\{<-\sqrt{2}, 1>\}}.$

Ex (Cont'd)

- Is $E_1 \perp E_2$?

↳ No! The bases aren't $\perp$!

$$\langle \sqrt{2}, 1 \rangle \cdot \langle -\sqrt{2}, 1 \rangle = -2+1 = -1.$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} \rightsquigarrow (1-\lambda)^2 - 4 = 0$ [char eq].

Eigenvalues are: $\lambda^2 - 2\lambda + 1 - 4 = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = 0$

$\Rightarrow (\lambda - 3)(\lambda + 1) = 0$

$\Rightarrow \lambda = 3, \ \lambda = -1$

- $\lambda = 3$: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \xrightarrow[\text{REF}]{} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{x} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- $\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \xrightarrow[\text{REF}]{} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

↳ Let $W_1 = \text{span}\{\langle 1, 1 \rangle\}$ & $W_2 = \text{span}\{\langle -1, 1 \rangle\}$ be eigenspaces for $\lambda = 3$ & $\lambda = -1$ respectively.

Q: $W_1 \perp W_2$? Yes

$$\langle -1, 1 \rangle \cdot \langle 1, 1 \rangle = -1+1 = 0!$$

- what is the difference?

↳ In ex 2, A is <u>symmetric</u>; in ex 1, it's not!

<u>Def</u>: An $n \times n$ matrix is <u>symmetric</u> if $A = A^T$.

<u>Theorem</u>: If A is symmetric & $n \times n$, then:

- A has n <u>real</u> eigenvalues (incl. multiplicities)
- The eigenspaces of A are mutually orthogonal.

<u>Ex</u>: Find eigenvectors for ($\lambda = -4, 4, 7$)

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 1 & 5 & 1 \\ 5 & 1 & 1 \end{pmatrix}$$ & show they're $\perp$!

~~[illegible]~~ $\hookrightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 5 \\ 1 & 5-\lambda & 1 \\ 5 & 1 & 1-\lambda \end{pmatrix}$

$\rightsquigarrow$ char poly