Chapter 6 stuff

Recall: • If $\vec{u} = \langle u_1, \dots, u_n \rangle$ & $\vec{v} = \langle v_1, \dots, v_n \rangle$, the dot prod. is $\vec{u} \cdot \vec{v} = u_1 v_1 + \dots + u_n v_n$

↳ This is aka the inner product of $\vec{u}$ & $\vec{v}$, which is written $\langle \vec{u}, \vec{v} \rangle$.

• Having an inner product allows us to ~~[illegible]~~ find distances, lengths, & angles:

- The length (or norm) of $\vec{v}$ is $\|\vec{v}\| \overset{\text{def}}{=} \sqrt{\vec{v} \cdot \vec{v}}$
- The distance between $\vec{u}$ & $\vec{v}$ is
  $$\text{dist}(\vec{u}, \vec{v}) \overset{\text{def}}{=} \|\vec{u} - \vec{v}\|$$
- The angle between $\vec{u}$ & $\vec{v}$ is θ s.t.
  $$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$
  sometimes write $\measuredangle(\vec{u}, \vec{v}) = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right)$

• **Ex:** Let $\vec{u} = \langle 1, 0, 1 \rangle$, $\vec{v} = \langle 2, 2, 1 \rangle$. Find:

- $\vec{u} \cdot \vec{v} = 2 + 0 + 1 = 3$
- $\|\vec{u}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$; $\|\vec{v}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$
- $\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|\langle -1, -2, 0 \rangle\| = \sqrt{(-1)^2 + (-2)^2 + 0^2} = \sqrt{5}$

  Note: $\|\vec{v} - \vec{u}\| = \|\langle 1, 2, 0 \rangle\| = \sqrt{(1)^2 + (2)^2 + 0^2} = \sqrt{5}$, so order doesn't matter
- $\measuredangle(\vec{u}, \vec{v}) = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right) = \cos^{-1}\left(\frac{3}{3\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

π

- In linear algebra, we <u>really</u> care about <u>orthogonality</u> (i.e. being perpendicular)

<u>Fact:</u> $\vec{u}$ is orthogonal to $\vec{v}$ (written: $\vec{u} \perp \vec{v}$) iff $\vec{u} \cdot \vec{v} = 0$.

↳ This implies the pythagorean thm: $\vec{u} \perp \vec{v} \iff \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

<u>Def:</u> The <u>orthogonal complement</u> of a subspace W of $\mathbb{R}^n$ is the subspace $W^\perp$ of $\mathbb{R}^n$ consisting of all vectors which are orthogonal to W:

$$W^\perp = \{\vec{v} \in \mathbb{R}^n : \langle \vec{v}, \vec{w} \rangle = 0 \text{ for } \underline{\text{all}}\ \vec{w} \in W\}.$$

<u>Ex:</u> let $\vec{u} = \langle 1,0,0 \rangle$, $\vec{v} = \langle 0,2,0 \rangle \in \mathbb{R}^3$, & let $W = \operatorname{span}\{\vec{u}, \vec{v}\}$.

<u>Note:</u> W = xy-plane in $\mathbb{R}^3$. & $\vec{u}, \vec{v}$ are a basis for W. To find $W^\perp$, look for a vector that's $\perp$ to <u>every</u> vector in the basis for W. Here, $\langle 0, 0, \square \rangle$ is $\perp$ to $\vec{u}$ & $\vec{v}$ for all $\square$:

$$\langle 1,0,0 \rangle \cdot \langle 0,0,\square \rangle = 0 \quad \underline{\text{and}} \quad \langle 0,2,0 \rangle \cdot \langle 0,0,\square \rangle = 0.$$

So, $W^\perp = \operatorname{span}\{\langle 0,0,1 \rangle\}$.

<u>Observe:</u>
- The only vector in W <u>and</u> $W^\perp$ is $\vec{0}$.
- If W is a subspace of $\mathbb{R}^n$, then $\dim(W) + \dim(W^\perp) = n$.

- Now, we want to put this together w/ stuff from Ch 4!

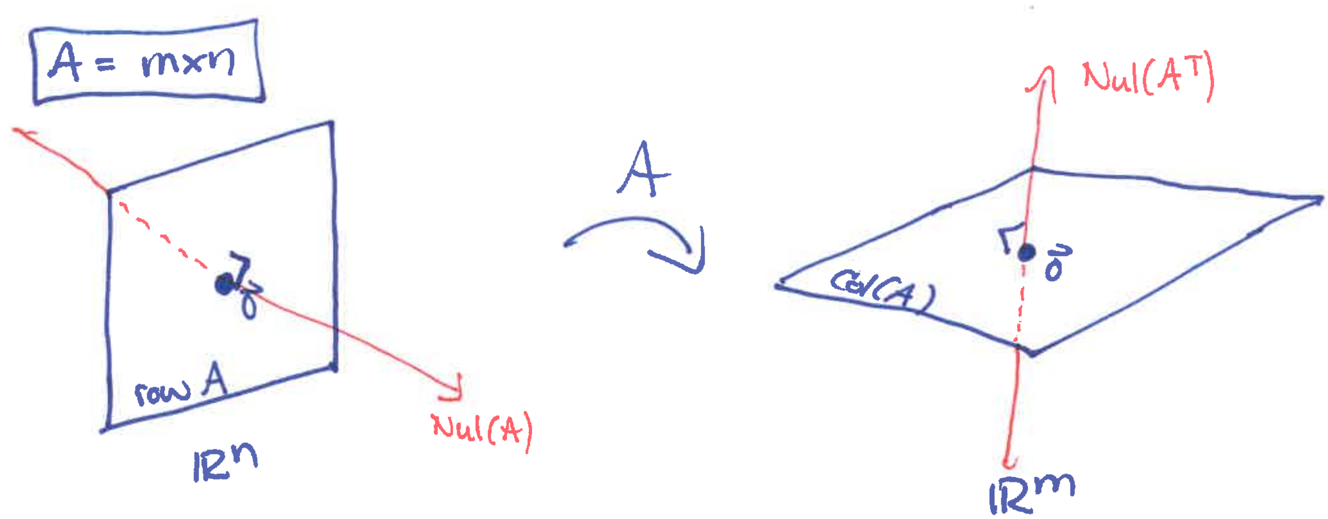
Theorem: For any matrix A,

Logic: use ① w/ A^T: $(Row(A^T))^\perp = nul(A^T)$
$\Leftrightarrow$ $(Col(A))^\perp = nul(A^T)$

$$\text{①}\ \underbrace{(Row\ A)^\perp = Nul(A)}\quad \&\quad (Col\ A)^\perp = Nul(A^T).$$

Logic: $\vec{x} \in Nul(A) \Leftrightarrow A\vec{x} = \vec{0}$, but components of $A\vec{x}$ are (rows of A)$\cdot\vec{x}$ ...

Here's one pic; for the other, see our webpage!



Ex: (From old class notes) [Found Nul(A) = span$\{\langle -3,-2,1,1,0\rangle, \langle -9/4,-4,7/4,0,1\rangle\}$]

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & -1 & -2 & 6 & 4 \\ -1 & 1 & 1 & -2 & 0 \\ 2 & 4 & 6 & 8 & 10 \end{pmatrix} \xrightarrow[\text{r.e.}]{} \begin{pmatrix} 1 & 0 & 0 & 3 & 9/4 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -1 & -7/4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $row(A) = span\{\overbrace{\langle 1,0,0,3,9/4\rangle}^{s_1}, \overbrace{\langle 0,1,0,2,4\rangle}^{s_2}, \overbrace{\langle 0,0,1,-1,-7/4\rangle}^{s_3}\}$ is a 3D subspace of $\mathbb{R}^5$.

  $\hookrightarrow$ $(row(A))^\perp$ is a 2D subspace w/ basis (e.g.)

$$\{\underbrace{\langle -3,-2,1,1,0\rangle}_{u}, \underbrace{\langle -9/4,-4,7/4,0,1\rangle}_{v}\}$$

Check: $u\cdot s_1=0$, $u\cdot s_2=0$, $u\cdot s_3=0$
$v\cdot s_1=0$, $v\cdot s_2=0$, $v\cdot s_3=0$

$\longrightarrow$

Ex (Cont'd)

- using RREF(A),

$$\text{Basis for } \underbrace{\text{Col}(A)}_{\swarrow} = \left\{ \overbrace{\langle 1,2,-1,2\rangle}^{a_1}, \overbrace{\langle 2,-1,1,4\rangle}^{a_2}, \overbrace{\langle 3,-2,1,6\rangle}^{a_3} \right\}$$

3D subspace of $\mathbb{R}^4$. $\Rightarrow$ $\text{nul}(A^T)$ has dim $= 1$

- Can find basis for $\text{nul}(A^T)$ by taking A^T & solving $(A^T)\vec{x} = \vec{0}$ (easy but long), or ...

- ... by finding one vector $\langle v_1, v_2, v_3, v_4\rangle$ orthogonal to $a_1, a_2,$ <u>and</u> a_3 (short in theory but hard...)

Long way: $A^T = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 4 \\ 3 & -2 & 1 & 6 \\ 4 & 6 & -2 & 8 \\ 5 & 4 & 0 & 10 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so

$$A^T\vec{x} = \vec{0} \Rightarrow \vec{x} = x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Nul}(A^T) = \text{span}\left\{ \langle -2,0,0,1\rangle \right\}.$$

check: $\langle -2,0,0,1\rangle \cdot a_1 = 0$

$\langle -2,0,0,1\rangle \cdot a_2 = 0$

$\langle -2,0,0,1\rangle \cdot a_3 = 0$