

§ 5.4 - The Matrix for a linear transform

This is a continuation of basis/coord. stuff from Ch 4, regardless of its section #.

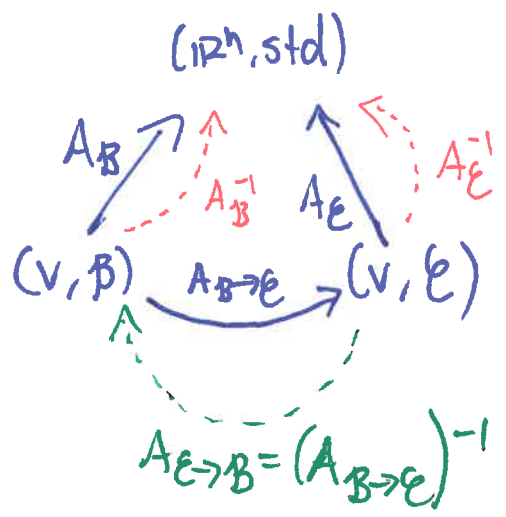
Recall: • Given basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for n -dim v.s. V , :

$$\left. \begin{array}{l} (\mathbb{R}^n, \text{std}) \\ \uparrow A_{\mathcal{B}} \\ (V, \mathcal{B}) \end{array} \right\} \Rightarrow \vec{x} = A_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} \text{ where } A_{\mathcal{B}} = [\vec{b}_1 | \dots | \vec{b}_n]$$

$$\downarrow A_{\mathcal{B}}^{-1}$$

$$A_{\mathcal{B}}^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}$$

• Given two bases \mathcal{B}, \mathcal{E} for n -dim v.s. V , :



$$[\vec{x}]_{\mathcal{E}} = A_{\mathcal{B} \rightarrow \mathcal{E}} [\vec{x}]_{\mathcal{B}}$$

$$(A_{\mathcal{B} \rightarrow \mathcal{E}})^{-1} [\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$$

where $A_{\mathcal{B} \rightarrow \mathcal{E}} = [[[\vec{b}_1]_{\mathcal{E}}] | \dots | [[\vec{b}_n]_{\mathcal{E}}]]$ and $= A_{\mathcal{E}}^{-1} A_{\mathcal{B}}$

$$A_{\mathcal{E} \rightarrow \mathcal{B}} = (A_{\mathcal{B} \rightarrow \mathcal{E}})^{-1} = A_{\mathcal{B}}^{-1} A_{\mathcal{E}}$$

• Recall: If $(V, \mathcal{E}) = (\mathbb{R}^n, \text{std})$, then $A_{\mathcal{B} \rightarrow \mathcal{E}} = A_{\mathcal{B}}$ from above!

New situation: Given a linear transformation $T: V \rightarrow V$, how does it "act" w.r.t different bases? What about transforms $T: V \rightarrow W$ between different v.s.'s / different bases?

Ex: let $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be a basis for \mathbb{R}^2 & consider $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ s.t. $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ y-x \end{pmatrix}$.

(i) Find $T(\vec{v}_1)$ & corr. \mathcal{B} -coords. $\begin{pmatrix} 2 & 1 & \vdots & -1 \\ 1 & 1 & \vdots & 0 \end{pmatrix} \rightarrow \text{RREF}$
 $T(\vec{v}_1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \stackrel{\mathcal{B}}{=} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

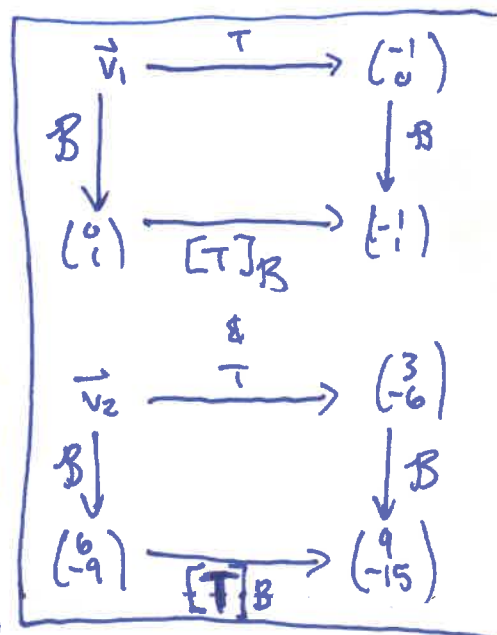
(ii) Find $T(\vec{v}_2)$ & corr. \mathcal{B} -coords. $\begin{pmatrix} 2 & 1 & \vdots & 3 \\ 1 & 1 & \vdots & -6 \end{pmatrix} \rightarrow \text{RREF}$
 $T(\vec{v}_2) = \begin{pmatrix} 3 \\ -6 \end{pmatrix} \stackrel{\mathcal{B}}{=} \begin{pmatrix} 9 \\ -15 \end{pmatrix}$

(iii) Find $T(\vec{b}_1)$ & corr \mathcal{B} -coords. $\begin{pmatrix} 2 & 1 & \vdots & -1 \\ 1 & 1 & \vdots & -1 \end{pmatrix} \rightarrow \text{RREF}$
 $T(\vec{b}_1) = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \stackrel{\mathcal{B}}{=} \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(iv) Find $T(\vec{b}_2)$ & corr \mathcal{B} coords. $\begin{pmatrix} 2 & 1 & \vdots & -1 \\ 1 & 1 & \vdots & 1 \end{pmatrix}$
 $T(\vec{b}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \stackrel{\mathcal{B}}{=} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Observe: • $[\vec{v}_1]_{\mathcal{B}} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
 $[\vec{v}_2]_{\mathcal{B}} = \left[\begin{pmatrix} 3 \\ -3 \end{pmatrix} \right]_{\mathcal{B}} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}$

• let $[T]_{\mathcal{B}} = \left[[T(\vec{b}_1)]_{\mathcal{B}} \mid [T(\vec{b}_2)]_{\mathcal{B}} \right]$
 $= \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$



• Then, $[T]_{\mathcal{B}} [\vec{v}_1]_{\mathcal{B}} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = [T(\vec{v}_1)]_{\mathcal{B}}$ by (i)
 $[T]_{\mathcal{B}} [\vec{v}_2]_{\mathcal{B}} = \dots \begin{pmatrix} 6 \\ -9 \end{pmatrix} = \begin{pmatrix} 9 \\ -15 \end{pmatrix} = [T(\vec{v}_2)]_{\mathcal{B}}$ by (ii)

So, what we've observed: If \mathcal{B} is a basis for \mathbb{R}^n (or n -dim vs V), then I can:

① Apply T to $(\mathbb{R}^n, \text{std})$, then write its \mathcal{B} -coords;

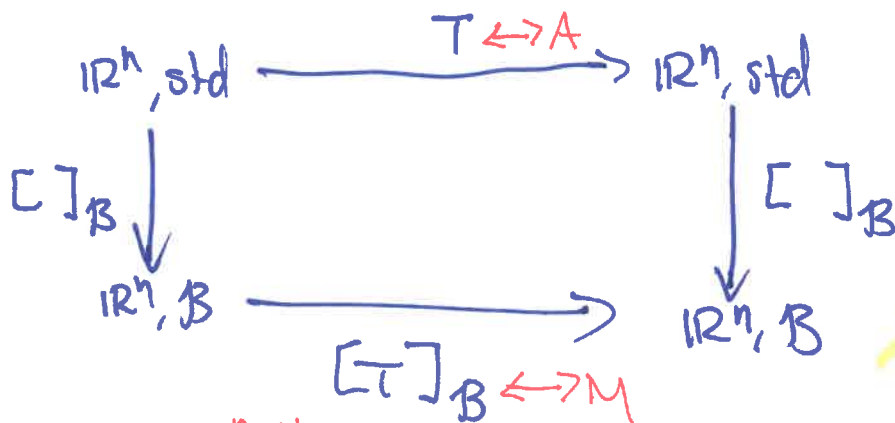
or

② Write its \mathcal{B} -coords, then apply $[T]_{\mathcal{B}}$

~~\mathcal{B} -coords of $T(\vec{v})$ for $\vec{v} \in \mathbb{R}^n$~~

$$[T]_{\mathcal{B}}^{\text{def}} = \left[[T(\vec{b}_1)]_{\mathcal{B}} \mid [T(\vec{b}_2)]_{\mathcal{B}} \right]$$

And the answer never changes! AKA, the following commutes.



$A = \text{canonical mat for } T$

$$\begin{array}{cccc}
 [T(\vec{x})]_{\mathcal{B}} & = & [T]_{\mathcal{B}} & [x]_{\mathcal{B}} \\
 \uparrow & & \uparrow & \uparrow \\
 (!!) & & (!!) & (!!)
 \end{array}$$

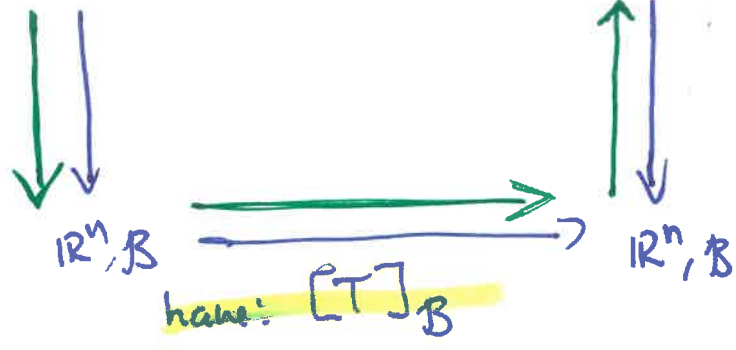
Ex: let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ be a basis for V vs V . Find $T(3\vec{b}_1 - 4\vec{b}_2)$ when $T: V \rightarrow V$ is linear trans whose matrix

rel. to \mathcal{B} is $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{pmatrix}$

- By previous diagram, there are two ways to do this: **Red path** and **green path**

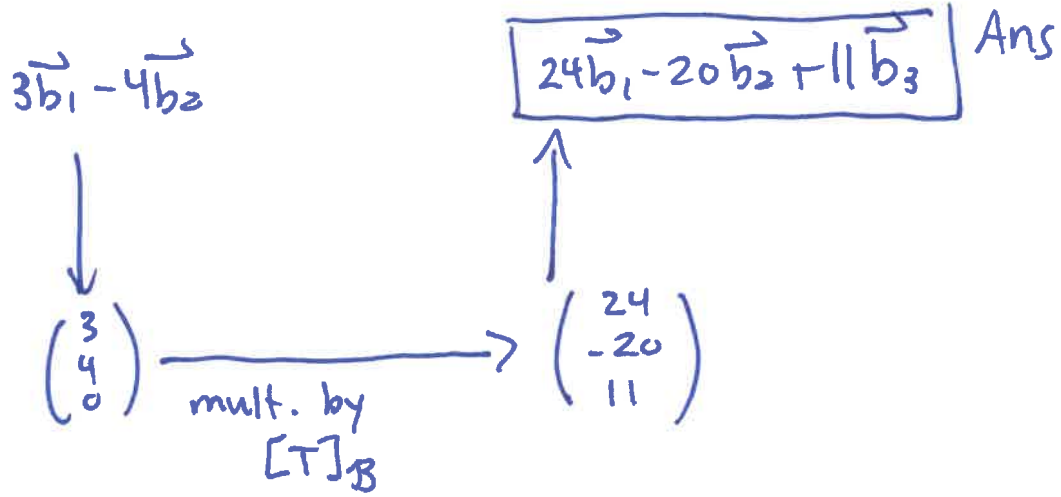
Ex (Cont'd)

Have: $3\vec{b}_1 - 4\vec{b}_2 \in \mathbb{R}^n, \text{std}$ $\xrightarrow{\text{red}}$ want: $T(3\vec{b}_1 - 4\vec{b}_2) \in \mathbb{R}^n, \text{std}$



Things we know

Green:



(longer path but way easier!)

Red: The problem is that we don't know $T!$ \Rightarrow have to build $T!$

$$\text{B/c } [T]_B = \begin{pmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{pmatrix} = \left[[T(\vec{b}_1)]_B \mid [T(\vec{b}_2)]_B \mid [T(\vec{b}_3)]_B \right]$$

we have:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = [T(\vec{b}_1)]_B \Rightarrow T(\vec{b}_1) = 0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3$$

$$\text{col 2} = [T(\vec{b}_2)]_B \Rightarrow T(\vec{b}_2) = -6\vec{b}_1 + 5\vec{b}_2 - 2\vec{b}_3$$

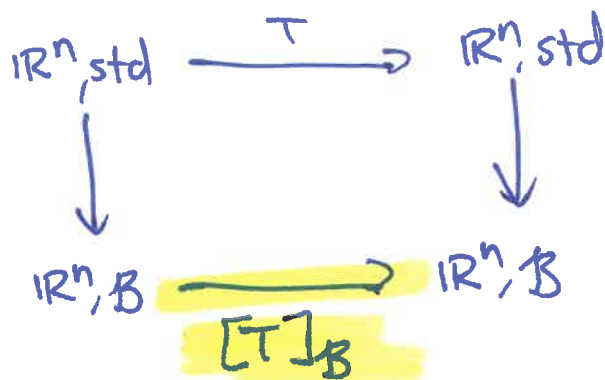
$$\text{col 3} = [T(\vec{b}_3)]_B \Rightarrow T(\vec{b}_3) = \vec{b}_1 - \vec{b}_2 + 7\vec{b}_3$$

Now, using linearity,

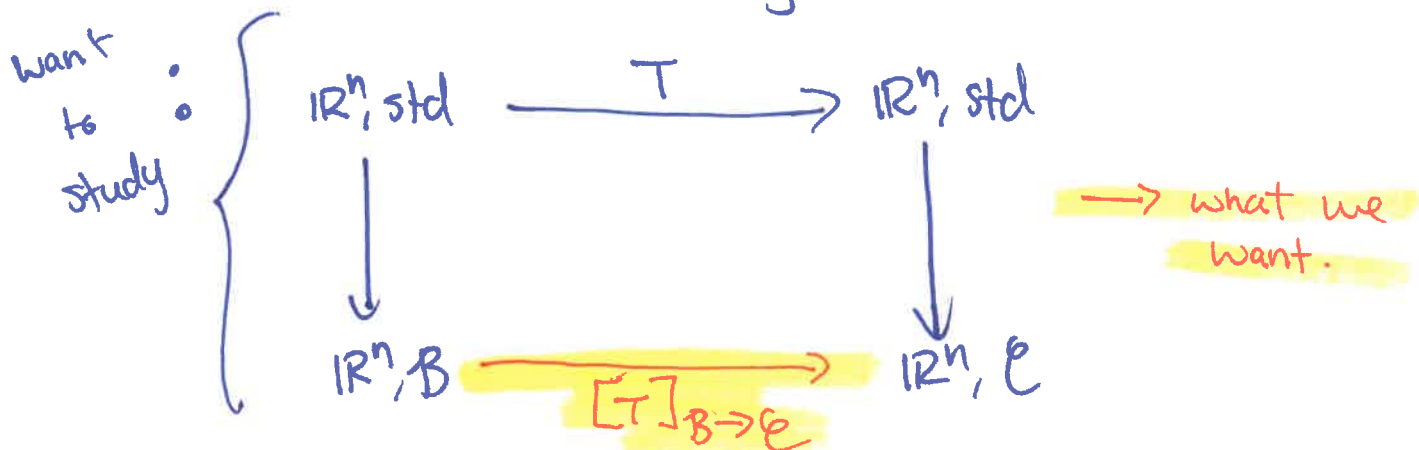
$$T(3\vec{b}_1 - 4\vec{b}_2) = 3T(\vec{b}_1) - 4T(\vec{b}_2) = 3(0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3) - 4(-6\vec{b}_1 + 5\vec{b}_2 - 2\vec{b}_3) = 24\vec{b}_1 - 20\vec{b}_2 + 11\vec{b}_3$$

- Those examples show how to perform the highlighted

map:



where both bottom spaces have same non-standard coords. What about when they don't?



Def: The desired matrix is

$$[T]_{B \rightarrow e} \stackrel{\text{def}}{=} [\![T(\vec{b}_1)]\!]_e \cdots [\![T(\vec{b}_n)]\!]_e$$

& satisfies: $[T(\vec{x})]_e = [T]_{B \rightarrow e} [\vec{x}]_B$.

note: If T is the identity, then $[id]_{B \rightarrow e} = A_{B \rightarrow e}$ from before!

Note: $\text{dom}(T) \neq \text{codom}(T)$ is fine! Replace one " \mathbb{R}^n " w/

" \mathbb{R}^m " in above & everything works, though

$[T]_{B \rightarrow e}$ won't be square then!

Ex. $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ basis for V & let $T: V \rightarrow W$
 let $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ basis for W

linear s.t. $T(\vec{b}_1) = 3\vec{c}_1 - 2\vec{c}_2 + 5\vec{c}_3$
 $T(\vec{b}_2) = 4\vec{c}_1 + 7\vec{c}_2 - \vec{c}_3$.

Find $[T]_{\mathcal{B} \rightarrow \mathcal{C}}$.

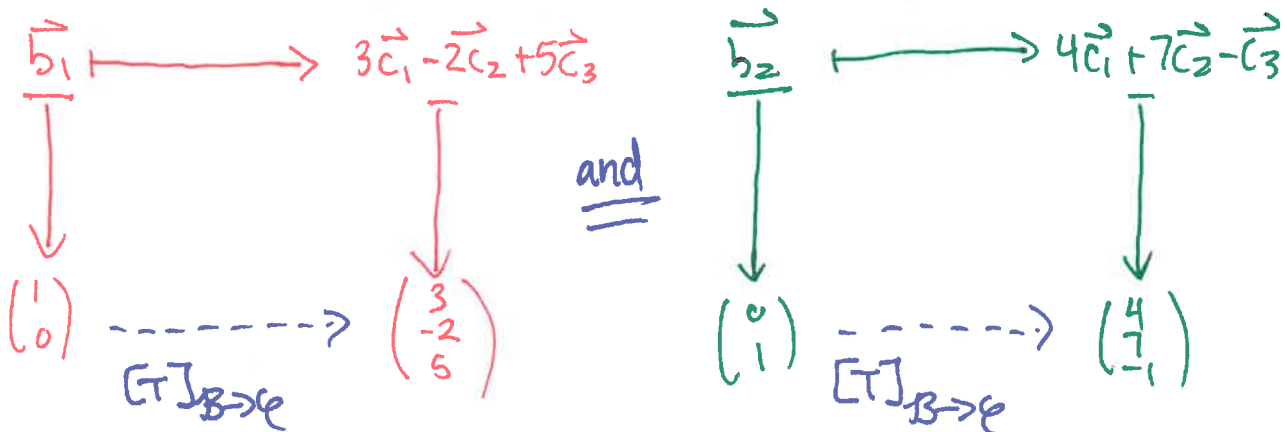
• From def,

$$[T]_{\mathcal{B} \rightarrow \mathcal{C}} = \left[[T(\vec{b}_1)]_{\mathcal{C}} \mid [T(\vec{b}_2)]_{\mathcal{C}} \right]$$

$$= \begin{pmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{pmatrix}. \quad \underline{\text{Ans}}$$

OR

• using the commutative diagram, we need the map $[T]_{\mathcal{B} \rightarrow \mathcal{C}}$ which sends $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix}$:



That means col 1 of $[T]_{\mathcal{B} \rightarrow \mathcal{C}}$ must be $\langle 3, -2, 5 \rangle^T$
 " " " " $\langle 4, 7, -1 \rangle^T$

$$\Rightarrow [T]_{\mathcal{B} \rightarrow \mathcal{C}} = \begin{pmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{pmatrix}. \quad \blacksquare$$