# §5.4 – The Matrix for a linear transform

This is a continuation of basis/coord. stuff from Ch 4, regardless of its section #.

**Recall:** • Given basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ for n-dim V.S. $V$, :

$$\begin{tikzcd}
(\mathbb{R}^n, \text{std}) \arrow[d, bend left, "A_{\mathcal{B}}^{-1}"] \\
(V, \mathcal{B}) \arrow[u, "A_{\mathcal{B}}"]
\end{tikzcd}
\quad \Rightarrow \quad
\begin{gathered}
\vec{x} = A_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} \text{ where } A_{\mathcal{B}} = [\vec{b}_1 | \cdots | \vec{b}_n] \\
\Updownarrow \\
A_{\mathcal{B}}^{-1} \vec{x} = [\vec{x}]_{\mathcal{B}}
\end{gathered}$$

• Given two bases $\mathcal{B}, \mathcal{E}$ for n-dim V.S. $V$, :

$$\begin{tikzcd}
 & (\mathbb{R}^n, \text{std}) \arrow[dl, dashed, bend right, "A_{\mathcal{B}}^{-1}"'] \arrow[dr, dashed, bend left, "A_{\mathcal{E}}^{-1}"] & \\
(V, \mathcal{B}) \arrow[ur, "A_{\mathcal{B}}"] \arrow[rr, bend right, "A_{\mathcal{B}\to\mathcal{E}}"] & & (V, \mathcal{E}) \arrow[ul, "A_{\mathcal{E}}"'] \arrow[ll, dashed, bend left=60, "A_{\mathcal{E}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{E}})^{-1}"]
\end{tikzcd}$$

$$[\vec{x}]_{\mathcal{E}} = A_{\mathcal{B}\to\mathcal{E}} [\vec{x}]_{\mathcal{B}}$$

$$\Updownarrow$$

$$(A_{\mathcal{B}\to\mathcal{E}})^{-1} [\vec{x}]_{\mathcal{E}} = [\vec{x}]_{\mathcal{B}}$$

where $A_{\mathcal{B}\to\mathcal{E}} = \big[ [\vec{b}_1]_{\mathcal{E}} | \cdots | [\vec{b}_n]_{\mathcal{E}} \big]$
$= A_{\mathcal{E}}^{-1} A_{\mathcal{B}}$ and

$A_{\mathcal{E}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{E}})^{-1} = A_{\mathcal{B}}^{-1} A_{\mathcal{E}}$.

• Recall: If $(V, \mathcal{E}) = (\mathbb{R}^n, \text{std})$, then $A_{\mathcal{B}\to\mathcal{E}} = A_{\mathcal{B}}$ from above!

**New situation:** Given a linear transformation $T: V \to V$, how does it "act" WRT different bases? What about transforms $T: V \to W$ between different V.S.'s / different bases?

<u>Ex:</u> let $\mathcal{B} = \left\{ \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ be a basis for $\mathbb{R}^2$ &

consider $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}-y\\y-x\end{pmatrix}$.

(i) Find $T\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\vec{v}_1}$ & corr. $\mathcal{B}$-coords. $\left(\begin{array}{cc|c}2&1&-1\\1&1&0\end{array}\right) \to$ RREF

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \overset{\mathcal{B}}{=\!=} \begin{pmatrix}-1\\1\end{pmatrix}$$

(ii) Find $T\underbrace{\begin{pmatrix}3\\-3\end{pmatrix}}_{\vec{v}_2}$ & corr. $\mathcal{B}$-coords. $\left(\begin{array}{cc|c}2&1&3\\1&1&-6\end{array}\right) \to$ RREF

$$T\begin{pmatrix}3\\-3\end{pmatrix} = \begin{pmatrix}3\\-6\end{pmatrix} \overset{\mathcal{B}}{=\!=} \begin{pmatrix}9\\-15\end{pmatrix}$$

(iii) Find $T(\vec{b}_1)$ & corr $\mathcal{B}$-coords. $\left(\begin{array}{cc|c}2&1&-1\\1&1&-1\end{array}\right) \to$ RREF

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} \overset{\mathcal{B}}{=\!=} \begin{pmatrix}0\\-1\end{pmatrix}$$

(iv) Find $T(\vec{b}_2)$ & corr $\mathcal{B}$ coords. $\left(\begin{array}{cc|c}2&1&-1\\1&1&1\end{array}\right)$

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix} \overset{\mathcal{B}}{\longleftrightarrow} \begin{pmatrix}-1\\1\end{pmatrix}$$

<u>Observe:</u>

- $[\vec{v}_1]_{\mathcal{B}} = \left[\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}0\\1\end{pmatrix}$

  $[\vec{v}_2]_{\mathcal{B}} = \left[\begin{pmatrix}3\\-3\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}6\\-9\end{pmatrix}$

- let $[T]_{\mathcal{B}} = \Big[ [T(\vec{b}_1)]_{\mathcal{B}} \,\Big|\, [T(\vec{b}_2)]_{\mathcal{B}} \Big]$

  $= \begin{bmatrix}0 & -1\\ -1 & 1\end{bmatrix}$

$$\begin{tikzcd}
\vec{v}_1 \arrow[r, "T"] \arrow[d, "\mathcal{B}"'] & \begin{pmatrix}-1\\0\end{pmatrix} \arrow[d, "\mathcal{B}"] \\
\begin{pmatrix}0\\1\end{pmatrix} \arrow[r, "{[T]_{\mathcal{B}}}"'] & \begin{pmatrix}-1\\1\end{pmatrix}
\end{tikzcd}$$

&

$$\begin{tikzcd}
\vec{v}_2 \arrow[r, "T"] \arrow[d, "\mathcal{B}"'] & \begin{pmatrix}3\\-6\end{pmatrix} \arrow[d, "\mathcal{B}"] \\
\begin{pmatrix}6\\-9\end{pmatrix} \arrow[r, "{[T]_{\mathcal{B}}}"'] & \begin{pmatrix}9\\-15\end{pmatrix}
\end{tikzcd}$$

- Then, $[T]_{\mathcal{B}}[\vec{v}_1]_{\mathcal{B}} = \begin{pmatrix}0 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = [T(\vec{v}_1)]_{\mathcal{B}}$ by (i)

  $[T]_{\mathcal{B}}[\vec{v}_2]_{\mathcal{B}} = \cdots \begin{pmatrix}6\\-9\end{pmatrix} = \begin{pmatrix}9\\-15\end{pmatrix} = [T(\vec{v}_2)]_{\mathcal{B}}$ by (ii)

So, what we've observed: If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ (or n-dim VS $V$), then I can:

① Apply $T$ to $(\mathbb{R}^n, \text{std})$, then write its $\mathcal{B}$-coords;

or

② Write its $\mathcal{B}$-coords, then apply $[T]_{\mathcal{B}}$:

$$[T]_{\mathcal{B}} \overset{\text{def}}{=} \left[\, [T(\vec{b}_1)]_{\mathcal{B}} \;\middle|\; [T(\vec{b}_2)]_{\mathcal{B}} \,\right].$$

And the answer never changes! AKA, the following commutes:

$$\begin{tikzcd}
\mathbb{R}^n, \text{std} \arrow[r, "T \leftrightarrow A"] \arrow[d, "{[\;]_{\mathcal{B}}}"'] & \mathbb{R}^n, \text{std} \arrow[d, "{[\;]_{\mathcal{B}}}"] \\
\mathbb{R}^n, \mathcal{B} \arrow[r, "{[T]_{\mathcal{B}} \leftrightarrow M}"'] & \mathbb{R}^n, \mathcal{B}
\end{tikzcd}$$

$A$ = canonical mat for $T$

$$[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}$$

(!!) (!!) (!!) (!!)

Ex: Let $\mathcal{B} = \underbrace{\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}}_{\text{in } \mathbb{R}^n, \text{ std}}$ be a basis for VS $V$. Find $\underbrace{T(3\vec{b}_1 - 4\vec{b}_2)}_{\text{Also in } \mathbb{R}^n, \text{ std}}$ when $T: V \to V$ is linear trans whose matrix rel. to $\mathcal{B}$ is $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{pmatrix}$

- By previous diagram, there are two ways to do this: Red path and green path

# Ex (Cont'd)

$$\begin{tikzcd}
\text{Have: } 3\vec{b}_1 - 4\vec{b}_2 \in \mathbb{R}^n, \text{std} \arrow[r] \arrow[d] & \text{want: } T(3\vec{b}_1 - 4\vec{b}_2) \in \mathbb{R}^n \text{ std} \arrow[d, shift left] \\
\mathbb{R}^n, \mathcal{B} \arrow[r, "\text{have: } {[T]_{\mathcal{B}}}"'] & \mathbb{R}^n, \mathcal{B} \arrow[u, shift left]
\end{tikzcd}$$

Things we know

Green:

(Longer path but way easier!)

$$\begin{tikzcd}
3\vec{b}_1 - 4\vec{b}_2 \arrow[d] & \boxed{24\vec{b}_1 - 20\vec{b}_2 + 11\vec{b}_3} \\
\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \arrow[r, "\text{mult. by } {[T]_{\mathcal{B}}}"'] & \begin{pmatrix} 24 \\ -20 \\ 11 \end{pmatrix} \arrow[u]
\end{tikzcd}$$

Ans

Red: The problem is that we don't know $T$! $\Rightarrow$ have to build $T$!

B/c $[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{pmatrix} = \Big[ [T(\vec{b}_1)]_{\mathcal{B}} \,\Big|\, [T(\vec{b}_2)]_{\mathcal{B}} \,\Big|\, [T(\vec{b}_3)]_{\mathcal{B}} \Big]$,

we have:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = [T(\vec{b}_1)]_{\mathcal{B}} \Rightarrow T(\vec{b}_1) = 0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3$$

$$\text{Col } 2 = [T(\vec{b}_2)]_{\mathcal{B}} \Rightarrow T(\vec{b}_2) = -6\vec{b}_1 + 5\vec{b}_2 - 2\vec{b}_3$$

$$\text{col } 3 = [T(\vec{b}_3)]_{\mathcal{B}} \Rightarrow T(\vec{b}_3) = \vec{b}_1 - \vec{b}_2 + 7\vec{b}_3.$$

Now, using linearity,

$$T(3\vec{b}_1 - 4\vec{b}_2) = 3T(\vec{b}_1) - 4T(\vec{b}_2) = \begin{matrix} 3(0\vec{b}_1 + 0\vec{b}_2 + \vec{b}_3) \\ -4(-6\vec{b}_1 + 5\vec{b}_2 - 2\vec{b}_3) \end{matrix} = \begin{matrix} 24\vec{b}_1 \\ -20\vec{b}_2 \\ +11\vec{b}_3. \end{matrix}$$

- Those examples show how to perform the highlighted map:

$$\begin{tikzcd} \mathbb{R}^n, \text{std} \arrow[r, "T"] \arrow[d] & \mathbb{R}^n, \text{std} \arrow[d] \\ \mathbb{R}^n, \mathcal{B} \arrow[r, "{[T]_{\mathcal{B}}}"'] & \mathbb{R}^n, \mathcal{B} \end{tikzcd}$$

where both bottom spaces have same non-standard coords. What about when they don't?

want to study:

$$\begin{tikzcd} \mathbb{R}^n, \text{std} \arrow[r, "T"] \arrow[d] & \mathbb{R}^n, \text{std} \arrow[d] \\ \mathbb{R}^n, \mathcal{B} \arrow[r, "{[T]_{\mathcal{B}\to\mathcal{C}}}"'] & \mathbb{R}^n, \mathcal{C} \end{tikzcd}$$

$\longrightarrow$ what we want.

<u>Def:</u> The desired matrix is

$$[T]_{\mathcal{B}\to\mathcal{C}} \overset{\text{def}}{=\!=} \Big[ [T(\vec{b}_1)]_{\mathcal{C}} \Big| \cdots \Big| [T(\vec{b}_n)]_{\mathcal{C}} \Big]$$

& satisfies: $[T(\vec{x})]_{\mathcal{C}} = [T]_{\mathcal{B}\to\mathcal{C}} [\vec{x}]_{\mathcal{B}}$.

<u>note:</u> If $T$ is the identity, then $[id]_{\mathcal{B}\to\mathcal{C}} = A_{\mathcal{B}\to\mathcal{C}}$ from before!

<u>Note:</u> $\text{dom}(T) \neq \text{codom}(T)$ is fine! Replace one "$\mathbb{R}^n$" w/ "$\mathbb{R}^m$" in above & everything works, though $[T]_{\mathcal{B}\to\mathcal{C}}$ won't be square then!

<u>Ex.</u> Let $\mathcal{B}=\{\vec{b}_1,\vec{b}_2\}$ basis for $V$ & let $T: V \to W$

$\mathcal{C}=\{\vec{c}_1,\vec{c}_2,\vec{c}_3\}$ basis for $W$

linear s.t. $T(\vec{b}_1)=3\vec{c}_1-2\vec{c}_2+5\vec{c}_3$

$T(\vec{b}_2)=4\vec{c}_1+7\vec{c}_2-\vec{c}_3$.

Find $[T]_{\mathcal{B}\to\mathcal{C}}$.

- From def,

$$[T]_{\mathcal{B}\to\mathcal{C}}=\Big[\,[T(\vec{b}_1)]_{\mathcal{C}}\ \Big|\ [T(\vec{b}_2)]_{\mathcal{C}}\,\Big]$$

$$=\begin{pmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{pmatrix}. \quad \underline{\text{Ans}}$$

**OR**

- using the commutative diagram, we need the map $[T]_{\mathcal{B}\to\mathcal{C}}$ which sends $\begin{pmatrix}1\\0\end{pmatrix} \longmapsto \begin{pmatrix}3\\-2\\5\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix} \longmapsto \begin{pmatrix}4\\7\\-1\end{pmatrix}$:

$$\begin{tikzcd}
\vec{b}_1 \arrow[r, mapsto] \arrow[d] & 3\vec{c}_1-2\vec{c}_2+5\vec{c}_3 \arrow[d] \\
\begin{pmatrix}1\\0\end{pmatrix} \arrow[r, dashed, "{[T]_{\mathcal{B}\to\mathcal{C}}}"'] & \begin{pmatrix}3\\-2\\5\end{pmatrix}
\end{tikzcd}
\quad \underline{\underline{\text{and}}} \quad
\begin{tikzcd}
\vec{b}_2 \arrow[r, mapsto] \arrow[d] & 4\vec{c}_1+7\vec{c}_2-\vec{c}_3 \arrow[d] \\
\begin{pmatrix}0\\1\end{pmatrix} \arrow[r, dashed, "{[T]_{\mathcal{B}\to\mathcal{C}}}"'] & \begin{pmatrix}4\\7\\-1\end{pmatrix}
\end{tikzcd}$$

That means col 1 of $[T]_{\mathcal{B}\to\mathcal{C}}$ must be $\langle 3,-2,5\rangle^T$

col 2 " " " " $\langle 4,7,-1\rangle^T$

$$\Rightarrow [T]_{\mathcal{B}\to\mathcal{C}}=\begin{pmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{pmatrix}. \quad \blacksquare$$