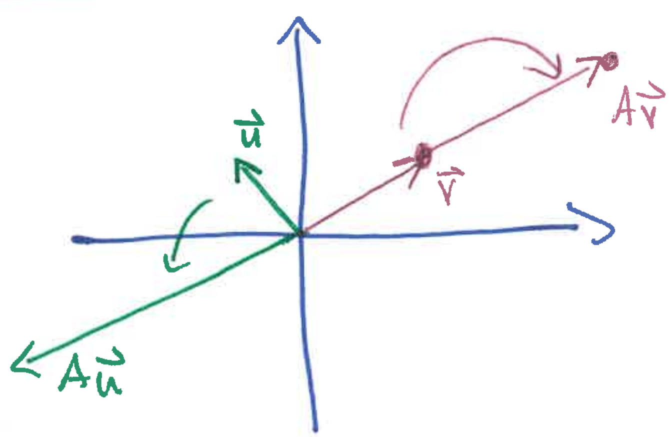
§ 5.1 & 5.2 - Eigenvalues, eigenvectors, eigenspaces

- Sometimes matrix multiplication is complicated; sometimes it's not.

Ex: let $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find $A\vec{u}$, $A\vec{v}$.

- $A\vec{u} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$
- $A\vec{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

↳ observe: $A\vec{u}$ not a scalar multiple of $\vec{u}$ but $A\vec{v} = 2\vec{v}$ is a scalar multiple of $\vec{v}$!



Def: A vector $\vec{x} \neq \vec{0}$ is said to be an eigenvector of an $n \times n$ matrix A if $A\vec{x} = \lambda \vec{x}$ for some constant λ. The scalar λ is called an eigenvalue of A.

↳ • In above example, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ w/ corresponding eigenvalue $\lambda = 2$. (b/c $A\vec{v} = 2\vec{v}$)

Ex: Is $\langle 1,-2,1\rangle^T$ an eigen vector of $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}$? If so, what is its corresponding eigenvalue?

Ans $\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

So it <u>is</u> an eigenvector & its corresponding eigenvalue is $\lambda = -2$.

Ex: Show that 7 is an eigenvalue of . Find the corresponding eigenvector.

$\hookrightarrow A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$

Ans • 7 is an eigenval for $A \iff A\vec{x} = 7\vec{x}$ for some $\vec{x} \neq \vec{0}$

$\iff A\vec{x} - 7\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$

$\iff (A - 7I)\vec{x} = \vec{0}$ " " ".

Now, $A - 7I = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$.

Note that cols are L.D. $\Rightarrow (A - 7I)\vec{x} = \vec{0}$ has a nontrivial solution by invertible matrix theorem. So 7 <u>is</u> an eigenvalue!

• To find eigenvectors: Put $(A - 7I \,\vdots\, \vec{0})$ into RREF:

$$(A - 7I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Now, as equations: $\begin{array}{l} x_1 - x_2 = 0 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2$. This is a whole <u>family</u> of vectors, and each one w/ $x_2 \neq 0$ is an eigenvec corresp. to $\lambda = 7$!

Note: In previous example, collection of eigenvalues was a subspace of $\mathbb{R}^2$. This is always the case!

Def: The set of <u>all solutions</u> of the eq. $(\overset{n\times n}{A}-\lambda I)\vec{x}=\vec{0}$ is a subspace of $\mathbb{R}^n$ called the eigenspace. Moreover:

$$\text{Eigen space}(A)=\left\{\begin{matrix}\text{all solutions of}\\ (A-\lambda I)\vec{x}=\vec{0}\end{matrix}\right\} = \underbrace{\text{nul}}_{\text{nullspace!}}(A-\lambda I)$$

Ex: $\lambda=2$ is an eigenvalue (e.val) for $A=\begin{pmatrix}4&-1&6\\2&1&6\\2&-1&8\end{pmatrix}$. Find the corresponding eigenspace.

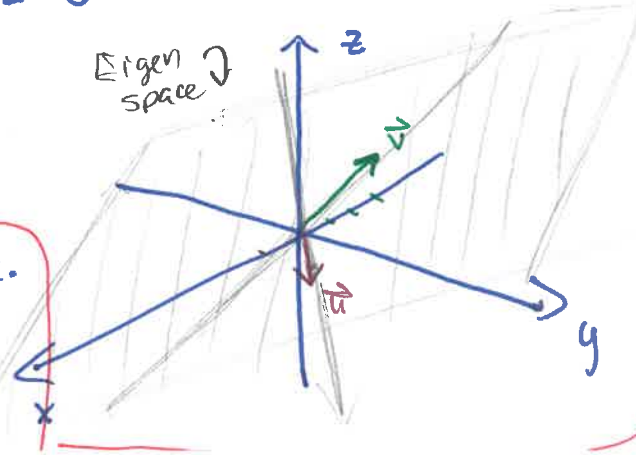
↳ • $\lambda=2$ is an e.val, so consider $A-2I$:

$$A-2I=\begin{pmatrix}2&-1&6\\2&-1&6\\2&-1&6\end{pmatrix}\xrightarrow[\text{REF}]{}\begin{pmatrix}2&-1&6\\0&0&0\\0&0&0\end{pmatrix}.$$

• As equations, augmenting w/ $\vec{0}$ yields

$$\left(\begin{array}{ccc|c}2&-1&6&0\\0&0&0&0\\0&0&0&0\end{array}\right)\to \begin{array}{l}2x_1=x_2-6x_3 \quad \text{divide by 2}\\ x_2=x_2+0x_3\\ x_3=0x_2+x_3\end{array} = x_2\begin{pmatrix}1/2\\1\\0\end{pmatrix}+x_3\begin{pmatrix}-3\\0\\1\end{pmatrix}.$$

• A basis is $\left\{\underset{\vec{u}}{\begin{pmatrix}1/2\\1\\0\end{pmatrix}},\underset{\vec{v}}{\begin{pmatrix}-3\\0\\1\end{pmatrix}}\right\}$.



• Meaning: Pick any vec $\vec{a}$ in eigenspace. (ex: $\vec{a}=\langle -2,2,1\rangle^T$) Then $A\vec{a}$ is $2\vec{a}$:

$$A\begin{pmatrix}-2\\2\\1\end{pmatrix}=\begin{pmatrix}-8-2+6\\-4+2+6\\-4-2+8\end{pmatrix}=\begin{pmatrix}-4\\4\\2\end{pmatrix}=2\vec{a}$$

Meaning (cont'd)

Any vec $\vec{b}$ not on eig. space won't satisfy $A\vec{b}=2\vec{b}$:

If $\vec{b}=\langle 1,1,1\rangle^T$, $A\vec{b}=\begin{pmatrix}9\\9\\9\end{pmatrix}\neq 2\vec{b}$.

Does equal $9\vec{b}$, so $\vec{b}$ on eigspace for $\lambda=9$!

- To comput eigenvals, we can:

{from invertible matrix theorem}

  - Consider $(A-\lambda I)\vec{x} = \vec{0}$

{only triv. soln $\vec{x}=\vec{0}$} $\swarrow$ $\searrow$ {nontrivial solns $\vec{x} \neq \vec{0}$}

$\det(A-\lambda I) \neq 0$ $\qquad$ $\det(A-\lambda I)=0$.

  - Note that the right path characterizes eigenvalues:

↳ $\boxed{\lambda \text{ eigenvalue of } A \iff \det(A-\lambda I)=0\,!}$

Def: The equation $\det(A-\lambda I)=0$ is called the <u>characteristic equation</u> of A.

Ex: Find char eq. of $A=\begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & +3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Recall: det (diag. matrix) = product of elts on the diagonal!

Ans: $\underset{\substack{\| \\ 0}}{\det(A-\lambda I)} \iff \det\begin{pmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = 0$

$\iff (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)=0$

$\iff \boxed{(5-\lambda)^2(3-\lambda)(1-\lambda)=0.}$ Char eq.

"characteristic polynomial"

In this ex., eigenvals of A are $\lambda=5, \lambda=3, \lambda=1$.

Ex: Find eigenvalues / vectors of

mention eval $=0$

(a) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \xleftrightarrow[\text{equiv}]{\text{row}}$ (b) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(a)

vals

$$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$= \lambda^2 - 2\lambda - 1$$

eigenvals different!

$$\Rightarrow \det(\cdots)=0 \Rightarrow \lambda^2-2\lambda-1=0$$

$$\Rightarrow \lambda = \frac{2\pm\sqrt{4+4}}{2}$$

$$\Rightarrow \lambda = 1\pm\sqrt{2}.$$

vecs

$$(A-\lambda I \mid \vec{0}) = \left(\begin{array}{cc|c} 2-\lambda & 1 & 0 \\ 1 & -\lambda & 0 \end{array}\right)$$

$\hookrightarrow$ • $\lambda = 1+\sqrt{2}$: $\left(\begin{array}{cc|c} 1-\sqrt{2} & 1 & 0 \\ 1 & -1-\sqrt{2} & 0 \end{array}\right)$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & -1-\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_1 = (1+\sqrt{2})x_2$

$\Rightarrow$ All vecs have form $x_2\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}$

• $\lambda = 1-\sqrt{2}$: similarly, all vecs have form $x_2\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$.

(b)

vals

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2$$

$$\Rightarrow \det(\cdots)=0 \Rightarrow \underbrace{\lambda=1,\ \lambda=1}_{\text{multiplicity } 2}$$

... we care!

vecs

$$(A-\lambda I \mid \vec{0}) = \left(\begin{array}{cc|c} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & 0 \end{array}\right)$$

$\hookrightarrow$ • $\lambda=1$: $\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right) \xrightarrow{\text{RREF}}$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \left.\begin{array}{l} x_1 = 0 \\ x_2 = \text{free} \end{array}\right]$ All eigenvecs have form $x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longleftrightarrow \lambda^2 + 1 = 0$

$\Rightarrow$ eigenvals are $\lambda = i$, $\lambda = -i$.

Eigenvecs:

- $\lambda = i$: $(A - \lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & -i & 0 \\ -i & -1 & 0 \end{array}\right) \xrightarrow{R_2 + iR_1 = R_2} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & \boxed{0} & 0 \end{array}\right)$

$\boxed{} = -1 + i(-i) = -1 - i^2 = -1 + 1 = 0$

$\Rightarrow \begin{array}{l} x_1 = i x_2 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = x_2 \boxed{\begin{pmatrix} i \\ 1 \end{pmatrix}}$ Basis for eigenspace for $\lambda = i$

- $\lambda = -i$: $(A - \lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right)$

$\xrightarrow{R_2 = R_2 - iR_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & \boxed{0} & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -i x_2 \\ x_2 = x_2 \end{array} = x_2 \boxed{\begin{pmatrix} -i \\ 1 \end{pmatrix}}$.

Basis for $\lambda = -i$ eigenspace

$\boxed{} = -1 - i(i) = -1 - i^2 = -1 - (-1) = 0$

So: $\lambda = i \longleftrightarrow \vec{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda = -i \longleftrightarrow \vec{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

Recall: The conjugate of $a+bi$ is $a-bi$. If $\begin{bmatrix} a+bi \\ c+di \end{bmatrix}$ is a vector, write as $\begin{bmatrix} a \\ c \end{bmatrix} + i \begin{bmatrix} b \\ d \end{bmatrix}$ & its conjugate is $\begin{bmatrix} a \\ c \end{bmatrix} - i \begin{bmatrix} b \\ d \end{bmatrix}$.

Ex (cont'd)

conjugates

Notice: $\lambda = i = 0 + 1i$ is an eigenval **AND** $\lambda = -i = 0 - i$ is an eigenval

basis for eigenspace $\begin{pmatrix} i \\ 1 \end{pmatrix}$ basis for eigenspace $\begin{pmatrix} -i \\ 1 \end{pmatrix}$

conjugates.

This is a general result !!

Notation: $\overline{a+bi} = a-bi$ (for conjugate)

Thm: If A is an $n \times n$ matrix w/ real entries, then any complex eigenvalues occur in conjugate pairs <u>and</u> the conjugate of an eigenvector is an eigenvector of the conjugate value:

$$\lambda = a+bi \text{ eigenval w/ vect } \vec{v} \implies \overline{\lambda} = a-bi \text{ eigenval w/ vect } \overline{\vec{v}}$$

Ex: Find eigenvals / basis for eigenspace for $\begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$

- char eq is $\det(A-\lambda I)=0 \iff (5-\lambda)(3-\lambda)+2=0$

$$\iff \lambda^2 - 8\lambda + 17 = 0$$

- eigenvals: $\lambda = \dfrac{8 \pm \sqrt{64-4(17)}}{2} = \dfrac{8 \pm \sqrt{-4}}{2} = \dfrac{8 \pm 2i}{2}$

$$= 1 \pm 4i.$$

- eigenvec for $\lambda = 1+4i$: $\left(\begin{array}{cc|c} 5-\lambda & -2 & 0 \\ 1 & 3-\lambda & 0 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 3-\lambda & 0 \\ 5-\lambda & -2 & 0 \end{array}\right)$

$$\xleftrightarrow{R_2 = R_2 - (5-\lambda)R_1} \left(\begin{array}{cc|c} 1 & 3-\lambda & 0 \\ 0 & \boxed{0} & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -(3-\lambda)x_2 \\ x_2 = x_2 \end{array}$$

$$\Downarrow$$

$2-(5-\lambda)(3-\lambda) = 0$

Basis $= \begin{pmatrix} \lambda-3 \\ 1 \end{pmatrix} = \begin{pmatrix} (1+4i)-3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4i-2 \\ 1 \end{pmatrix}$

- eigenvec for $1-4i$:

$\overline{\begin{pmatrix} -2+4i \\ 1 \end{pmatrix}} = \begin{pmatrix} -2-4i \\ 1 \end{pmatrix}.$

Theorem: If A is an $n \times n$ matrix w/ real entries, then A has n eigenvalues (counting multiplicities) which may be complex. Complex eigenvals come in conjugate pairs.

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