§ 4.7 – Change of Basis

In the last section, we learned how to convert coordinates for a vector in $\mathbb{R}^n$ (WRT std basis) to coords in an n-dim V.S. V w/ basis $\mathcal{B}$:

↳ If $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ and $A_{\mathcal{B}} = [\vec{b}_1 | \cdots | \vec{b}_n]$, then

$$\underbrace{\underset{\substack{\uparrow \\ \mathbb{R}^n \text{ w/} \\ \text{std basis}}}{\vec{x}} = A_{\mathcal{B}} \underset{\substack{\uparrow \\ V \text{ w/ } \mathcal{B} \\ \text{basis}}}{[\vec{x}]_{\mathcal{B}}}}_{\substack{(V, \mathcal{B}) \text{ coords into} \\ (\mathbb{R}^n, \text{std}) \text{ coords}}} \quad \& \quad \underbrace{[\vec{x}]_{\mathcal{B}} = A_{\mathcal{B}}^{-1} \vec{x}.}_{\substack{(\mathbb{R}^n, \text{std}) \text{ coords into} \\ (V, \mathcal{B}) \text{ coords}}}$$

Now, we want to tackle the related question:

↳ If V = n-dim V.S. & $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ are two bases
$\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_n\}$
for V, how do we convert $\mathcal{B}$-coords into $\mathcal{C}$-coords?] And vice versa

Ex Let V be a 2D V.S. & let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$, $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for V. Suppose [redacted]
[redacted] we know that

$$\vec{x} = 3\vec{b}_1 + \vec{b}_2, \quad \vec{b}_1 = 4\vec{c}_1 + \vec{c}_2, \text{ and } \vec{b}_2 = -6\vec{c}_1 + \vec{c}_2.$$

Find $[\vec{x}]_{\mathcal{C}}$. ⟶

Ex (Cont'd)

Sol'n # 1 — Substitute

$$\vec{x} = 3\underset{\substack{\uparrow \\ \vec{b}_1 = 4\vec{c}_1 + \vec{c}_2}}{\vec{b}_1} + \underset{\vec{b}_2 = -6\vec{c}_1 + \vec{c}_2}{\vec{b}_2} = 3(4\vec{c}_1 + \vec{c}_2) + (-6\vec{c}_1 + \vec{c}_2) = \begin{array}{l} 12\vec{c}_1 + 3\vec{c}_2 \\ -6\vec{c}_1 + \vec{c}_2 \end{array}$$

$$= 6\vec{c}_1 + 4\vec{c}_2$$

$$\Rightarrow \quad [\vec{x}]_{\mathcal{C}} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

Sol'n # 2 - Matrices

Want $\underbrace{[\vec{x}]_{\mathcal{C}} = [3\vec{b}_1 + \vec{b}_2]_{\mathcal{C}}}_{\vec{x} = 3\vec{b}_1 + \vec{b}_2}$. B/c coord. transform is linear,

$$= 3\,[\vec{b}_1]_{\mathcal{C}} + [\vec{b}_2]_{\mathcal{C}}$$

$$= \underbrace{\Big[[\vec{b}_1]_{\mathcal{C}} \;\Big|\; [\vec{b}_2] \Big]}_{\substack{\text{matrix w/} \\ [\vec{b}_1]_{\mathcal{C}} \ \& \ [\vec{b}_2]_{\mathcal{C}} \text{ as cols}}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

But we know $[\vec{b}_1]_{\mathcal{C}}$, $[\vec{b}_2]_{\mathcal{C}}$:

$$\Rightarrow [\vec{x}]_{\mathcal{C}} = \begin{pmatrix} 4 & \vdots & -6 \\ 1 & \vdots & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 - 6 \\ 3 + 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}. \ \blacksquare$$

=

The 2nd way has the benefit of being generalizable!

This is also good b/c it always works!

Theorem: If V is a v.s. w/ bases $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ & $\mathcal{C}=\{\vec{c}_1,\dots,\vec{c}_n\}$, then there exists a matrix $A_{\mathcal{B}\to\mathcal{C}}$ s.t.

$$[\vec{x}]_{\mathcal{C}} = A_{\mathcal{B}\to\mathcal{C}}[\vec{x}]_{\mathcal{B}} \quad \text{for all } \vec{x}\in V, \text{ AND}$$

It ALWAYS has the form

$A_{\mathcal{B}\to\mathcal{C}}$ = change of coord. matrix

$$A_{\mathcal{B}\to\mathcal{C}} = \left[\,[\vec{b}_1]_{\mathcal{C}} \;\vdots\; \cdots \;\vdots\; [\vec{b}_n]_{\mathcal{C}}\,\right]$$

← Need "old basis" coords relative to new basis!

Ex: $\mathcal{B}=\left\{\vec{b}_1=\begin{pmatrix}-9\\1\end{pmatrix}, \vec{b}_2=\begin{pmatrix}-5\\-1\end{pmatrix}\right\}$, $\mathcal{C}=\left\{\vec{c}_1=\begin{pmatrix}1\\-4\end{pmatrix}, \vec{c}_2=\begin{pmatrix}3\\-5\end{pmatrix}\right\}$

are two bases for $\mathbb{R}^2$.

soln: we know $A_{\mathcal{B}\to\mathcal{C}} = \left[\,[\vec{b}_1]_{\mathcal{C}} \;\vdots\; [\vec{b}_2]_{\mathcal{C}}\,\right]$. we need to know what those columns are!

↳ • let $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ & $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$ (want to find x_1, x_2, y_1, y_2 ...)

- By def, $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix}x_1\\x_2\end{bmatrix} \longleftrightarrow \vec{b}_1 = [\vec{c}_1 \vdots \vec{c}_2]\begin{pmatrix}x_1\\x_2\end{pmatrix}$ (*)

  $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix}y_1\\y_2\end{bmatrix} \longleftrightarrow \vec{b}_2 = [\vec{c}_1 \vdots \vec{c}_2]\begin{pmatrix}y_1\\y_2\end{pmatrix}$ (**)

- one way to solve (*) is to augment: $[\vec{c}_1 \vdots \vec{c}_2 \vdots \overset{\vec{b}_1}{\text{[illegible]}}]$

  (**) " " " : $[\vec{c}_1 \vdots \vec{c}_2 \vdots \underset{\vec{b}_2}{\text{[illegible]}}]$

  (***)

↳ B/c (***) is the same, we can augment both simultaneously & solve!

$$[\vec{c}_1 \vdots \vec{c}_2 \mid \vec{b}_1 \vdots \vec{b}_2] \xrightarrow[\text{RREF}]{} [I_n \mid A_{\mathcal{B}\to\mathcal{C}}]$$

Ex (Cont'd)

$$\left[\begin{array}{cc|cc}1&3&-9&-5\\-4&-5&1&-1\\ \vec{c}_1&\vec{c}_2&\vec{b}_1&\vec{b}_2\end{array}\right] \longrightarrow \left[\begin{array}{cc|cc}1&3&-9&-5\\0&7&-35&-21\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc}1&3&-9&-5\\0&1&-5&-3\end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|cc}1&0&6&4\\0&1&-5&-3\\ & &[\vec{b}_1]_{\mathcal{C}}&[\vec{b}_2]_{\mathcal{C}}\end{array}\right] \quad ■$$

$$\underbrace{\begin{matrix}6&4\\-5&-3\end{matrix}}_{A_{\mathcal{B}\to\mathcal{C}}}$$

Observe:

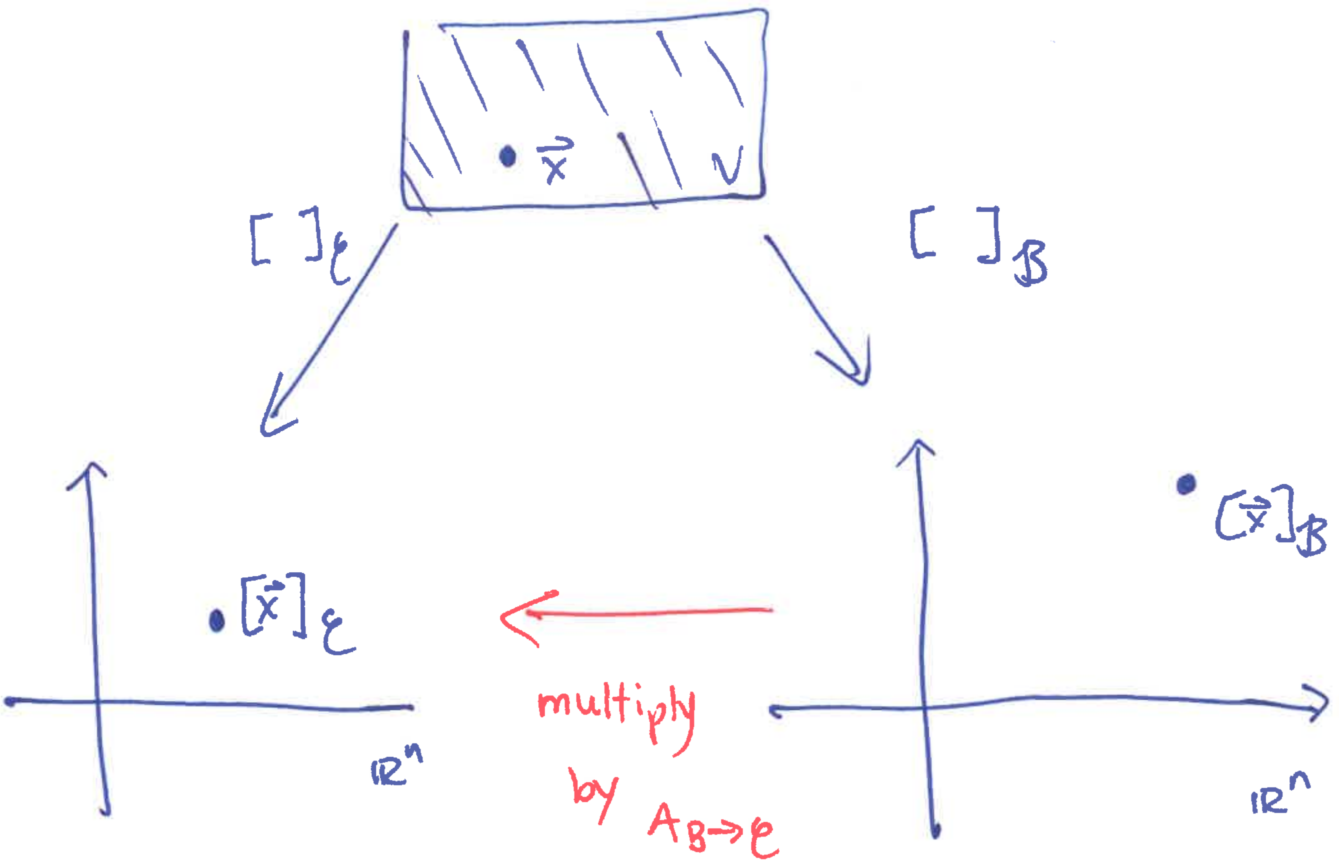
- If $\overbrace{\vec{x} = \begin{pmatrix}-9\\1\end{pmatrix}}^{\vec{x}=\vec{b}_1}$, then $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}1\\0\end{pmatrix}$ &

  $$A_{\mathcal{B}\to\mathcal{C}}\cdot[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}6&4\\-5&-3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}6\\-5\end{pmatrix} = [\vec{b}_1]_{\mathcal{C}}$$

- If $\overbrace{\vec{x} = \begin{pmatrix}-5\\-1\end{pmatrix}}^{\vec{x}=\vec{b}_2}$, then $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}0\\1\end{pmatrix}$ &

  $$A_{\mathcal{B}\to\mathcal{C}}[\vec{x}_2]_{\mathcal{B}} = \begin{pmatrix}6&4\\-5&-3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix} = [\vec{b}_2]_{\mathcal{C}}$$

Geometrically



Note: The columns of $A^{-}_{B\to C}$ are L.I. since they're $\mathcal{C}$-coord. vectors for a (L.I.) basis $\mathcal{B}$. Hence,

$$(A_{B\to C})^{-1} \text{ exists!}$$

Def: $(A_{B\to c})^{-1} = A_{C\to B}$ is the matrix taking a $\mathcal{C}$-coord to a $\mathcal{B}$-coord:

$$[x]_{\mathcal{C}} = A_{\mathcal{B}\to\mathcal{C}}[\vec{x}]_{\mathcal{B}} \iff A_{\mathcal{C}\to\mathcal{B}}[\vec{x}]_{\mathcal{C}} = [\vec{x}]_{\mathcal{B}}.$$

Ex: Find $A_{\mathcal{B}\to\mathcal{C}}$ & $A_{\mathcal{C}\to\mathcal{B}}$ for

$$\mathcal{B} = \left\{ \underbrace{\begin{pmatrix}-1\\8\end{pmatrix}}_{b_1}, \underbrace{\begin{pmatrix}1\\-5\end{pmatrix}}_{b_2} \right\} \qquad \mathcal{C} = \left\{ \underbrace{\begin{pmatrix}1\\4\end{pmatrix}}_{c_1}, \underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{c_2} \right\}$$

- $$A_{\mathcal{B}\to\mathcal{C}} = \Big[\underbrace{[\vec{b}_1]_{\mathcal{C}}}_{=\begin{pmatrix}x_1\\x_2\end{pmatrix}} \Big| \underbrace{[\vec{b}_2]_{\mathcal{C}}}_{=\begin{pmatrix}y_1\\y_2\end{pmatrix}} \Big] \longleftrightarrow \begin{array}{l} [\vec{c}_1\ \ \vec{c}_2 \,\vdots\, \overset{\vec{b}_1}{\cancel{\ \ }}] = \cancel{\ \ } \begin{pmatrix}x_1\\x_2\end{pmatrix} \\ [\vec{c}_1\ \ \vec{c}_2 \,\vdots\, \underset{\vec{b}_2}{\cancel{\ \ }}] = \cancel{\ \ } \begin{pmatrix}y_1\\y_2\end{pmatrix} \end{array}$$

$\Rightarrow$ can solve both:
$$\left[\begin{array}{cc:cc} 1 & 1 & -1 & 1 \\ 4 & 1 & 8 & -5 \end{array}\right] \xrightarrow[\text{RREF}]{} \left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & \underbrace{-4 \quad 3}_{A_{\mathcal{B}\to\mathcal{C}}} \end{array}\right]$$

- $$\left(A_{\mathcal{B}\to\mathcal{C}}\right)^{-1} = A_{\mathcal{C}\to\mathcal{B}}$$

$$= \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & \overbrace{2}^{[\vec{c}_2]_{\mathcal{B}}} \\ \underbrace{4}_{[\vec{c}_1]_{\mathcal{B}}} & 3 \end{pmatrix}$$

Check

$$\vec{x} = \begin{pmatrix}1\\4\end{pmatrix} \Rightarrow [\vec{x}]_{\mathcal{C}} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{pmatrix}3 & 2\\4 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

$$\vec{x} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow [\vec{x}]_{\mathcal{C}} = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{pmatrix}3 & 2\\4 & 3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$

- we can relate $A_{\mathcal{B}\to\mathcal{E}}$ & $A_{\mathcal{E}\to\mathcal{B}}$ into the matrices $A_{\mathcal{B}}$ & $A_{\mathcal{E}}$ from § 4.4:

$$A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x} \quad \& \quad A_{\mathcal{E}}[\vec{x}]_{\mathcal{E}} = \vec{x} \qquad \text{so}$$

$$\Rightarrow [\vec{x}]_{\mathcal{E}} = A_{\mathcal{E}}^{-1}\vec{x},$$

we can write

$$[\vec{x}]_{\mathcal{E}} = A_{\mathcal{B}\to\mathcal{E}}[\vec{x}]_{\mathcal{B}} \quad (\text{this section}) \quad \underline{\underline{\text{or}}}$$

$$[\vec{x}]_{\mathcal{E}} = A_{\mathcal{E}}^{-1}\vec{x} = A_{\mathcal{E}}^{-1}(A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}})$$

$$= A_{\mathcal{E}}^{-1}A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}.$$

So $A_{\mathcal{B}\to\mathcal{E}} = A_{\mathcal{E}}^{-1}A_{\mathcal{B}}$!

$$\begin{tikzcd}
 & \boxed{\mathbb{R}^n_{std}} \arrow[dl, bend right=20, "A_{\mathcal{B}}^{-1}"'] \arrow[dr, "A_{\mathcal{E}}^{-1}"] & \\
\boxed{\mathbb{R}^n_{\mathcal{B}}} \arrow[ur, bend right=20, "A_{\mathcal{B}}"'] \arrow[rr, "A_{\mathcal{B}\to\mathcal{E}}"'] & & \boxed{\mathbb{R}^n_{\mathcal{E}}} \arrow[ul, bend right=20, "A_{\mathcal{E}}"'] \arrow[ll, bend left=50, "A_{\mathcal{E}\to\mathcal{B}}"]
\end{tikzcd}$$

So, this diagram "commutes": Getting from any vertex ☐ to any other allows you to trace arrows in <u>any</u> order!

$\Rightarrow$ Getting $\mathbb{R}^n_{\mathcal{B}} \to \mathbb{R}^n_{\mathcal{E}}$ can be $A_{\mathcal{B}\to\mathcal{E}}$ $\underline{\underline{\text{or}}}$ $A_{\mathcal{E}}^{-1}A_{\mathcal{B}}$ b/c they're equal!

- Also:

$$A_{\mathcal{B}}: \mathbb{R}^n_{\mathcal{B}} \longrightarrow \mathbb{R}^n_{std}, \quad \text{so} \quad \text{if } \mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\},$$

$$A_{\mathcal{B} \Rightarrow std} = \left[[\vec{b}_1]_{std} \,\middle|\, \cdots \,\middle|\, [\vec{b}_n]_{std} \right]$$

$$= \left[\vec{b}_1 \,\middle|\, \cdots \,\middle|\, \vec{b}_n \right]$$

$$= A_{\mathcal{B}} \,!$$