§ 4.4 - Coordinate Systems

- From now on, I'll use the term "vector space" <u>without</u> lecturing on that:
  - ↳ ∘ For a definition, see the handout
    - ∘ In your mind, think "subspace of some $\mathbb{R}^n$" everytime you encounter that word!

Recall: A **basis** for an n-dim vector space V is a set $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ which is linearly independent & which satisfies $\text{span}\{\vec{b}_1, \dots, \vec{b}_n\} = V$.

↳ • B/c $\text{span}\{\vec{b}_1, \dots, \vec{b}_n\} = V$, every vector $\vec{x} \in V$ is a linear combo $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ of elements of $\mathcal{B}$. ← These $c_1, \dots, c_n$ are <u>unique</u>.

Ex: $\mathcal{B}$: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. (Check this!) WRT this basis, we can write $\begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{R}^2$:

$$\begin{pmatrix}0\\1\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\2\end{pmatrix} \Rightarrow \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c_1+c_2\\2c_2\end{pmatrix} \Rightarrow \begin{matrix}c_1 = -1/2\\ c_2 = 1/2\end{matrix}$$

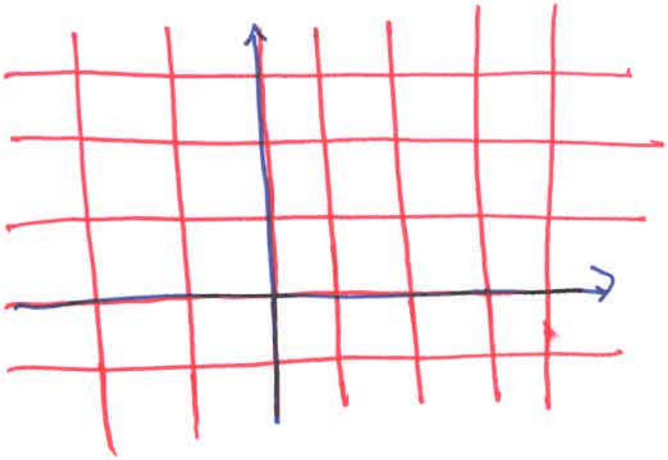
So $$\begin{pmatrix}0\\1\end{pmatrix} = \frac{-1}{2}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\2\end{pmatrix}!$$

- we can imagine these values c_1, c_2 as "coordinates" for $\begin{pmatrix}0\\1\end{pmatrix}$ WRT the basis $\mathcal{B}$:

$$\left[\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{B}} \overset{\text{def}}{=} \text{Coordinate vector of } \begin{pmatrix}0\\1\end{pmatrix} \text{ WRT basis } \mathcal{B}$$

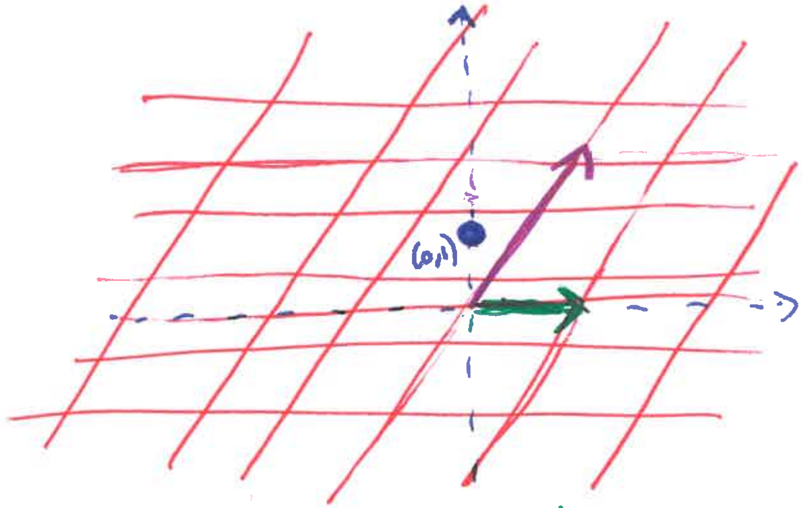
$$= \begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-1/2\\1/2\end{pmatrix}.$$

standard basis for $\mathbb{R}^2$



"old graphing paper"

Basis $\mathcal{B}$ for $\mathbb{R}^2$



"new graphing paper"

$= \begin{pmatrix}1\\0\end{pmatrix}$

$= \begin{pmatrix}1\\2\end{pmatrix}$

- $(0,1) = (-\frac{1}{2}, \frac{1}{2})$ WRT $\mathcal{B}$

So:

- If $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for v.s. V & $\vec{x} \in V$, there exists unique constants $c_1, \dots, c_n \in \mathbb{R}$ s.t. $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$.

- Def: The coordinates of $\vec{x}$ WRT $\mathcal{B}$ (aka $\mathcal{B}$-coords of $\vec{x}$) are the vals $c_1, \dots, c_n$

- Def: The coordinate vector of $\vec{x}$ WRT $\mathcal{B}$ (aka $\mathcal{B}$-coord vector) is the vector

$$[\vec{x}]_{\mathcal{B}} \overset{\text{def}}{=} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Ex: Find $[\vec{x}]_{\mathcal{B}}$ where $\mathcal{B} = \left\{ \begin{pmatrix}1\\-1\\-3\end{pmatrix}, \begin{pmatrix}-3\\4\\9\end{pmatrix}, \begin{pmatrix}2\\-2\\4\end{pmatrix} \right\}$ & $\vec{x} = \begin{pmatrix}8\\-9\\6\end{pmatrix}$ (*)

- want c_1, c_2, c_3 s.t. $\begin{pmatrix}8\\-9\\6\end{pmatrix} = c_1\begin{pmatrix}1\\-1\\-3\end{pmatrix} + c_2\begin{pmatrix}-3\\4\\9\end{pmatrix} + c_3\begin{pmatrix}2\\-2\\4\end{pmatrix}$

$$= \underbrace{\begin{pmatrix}1 & -3 & 2\\-1 & 4 & -2\\-3 & 9 & 4\end{pmatrix}}_{A}\underbrace{\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}}_{\vec{c}} = A\vec{c}$$

$\hookrightarrow$ want to solve $A\vec{c} = \vec{x}$.

Ex (Cont'd)

- In terms of augmented matrices,

$$\left(\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{array}\right) \xrightarrow{\text{REF}} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

$(A \mid \vec{x})$

$$\Rightarrow \left.\begin{array}{l} c_3 = 3 \\ c_2 = -1 \end{array}\right] \Rightarrow \begin{array}{l} c_1 - 3(-1) + 2(3) = 8 \\ \Rightarrow c_1 + 3 + 6 = 8 \\ \Rightarrow c_1 + 9 = 8 \Rightarrow c_1 = -1 \end{array}$$

So $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$. ∎

Note: This eq. (*) had the form $\vec{x} = A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$ where LHS is WRT standard basis & $[\vec{x}]_{\mathcal{B}}$ is WRT $\mathcal{B}$

- In the previous example, note that we could augment any vector $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$ to form $(A \mid \vec{y})$ and the result would be the weights c_1, c_2, c_3 needed to write $\vec{y}$ (WRT the standard basis) as $[\vec{y}]_{\mathcal{B}}$ (WRT the basis in that particular example.

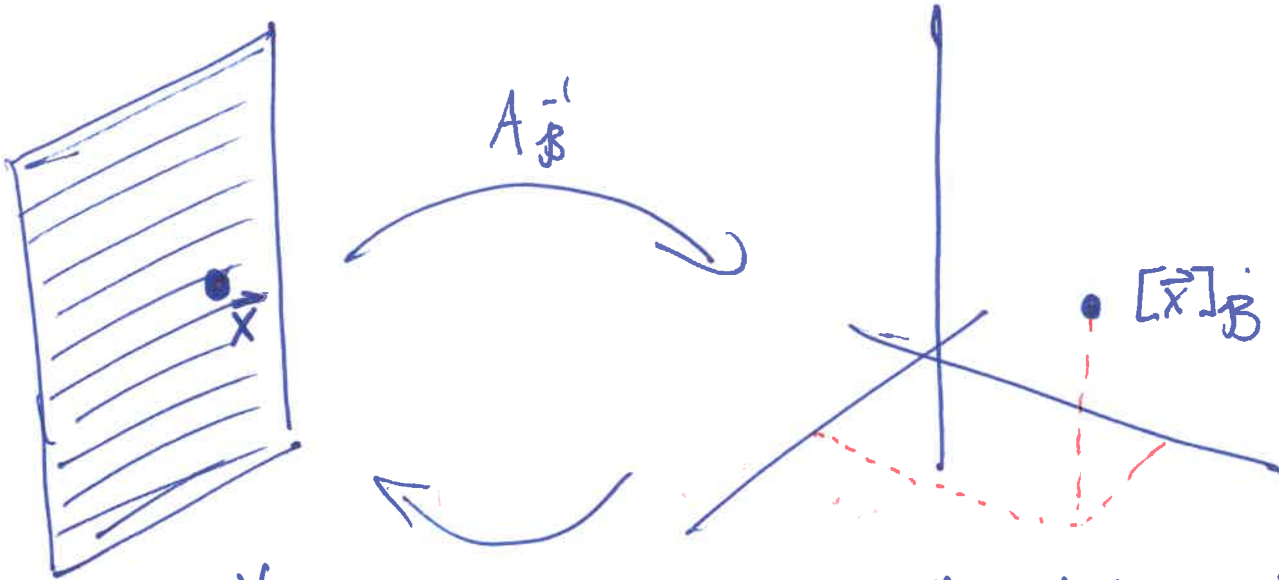
$$\mathcal{B} = \left\{ \vec{b}_1 = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}, \vec{b}_2 = \begin{pmatrix} -3 \\ 4 \\ 9 \end{pmatrix}, \vec{b}_3 = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} \right\}$$

This has a name:

Def: Given a basis for the n-dim V.S. V, The matrix $A_{\mathcal{B}} = [\vec{b}_1 | \cdots | \vec{b}_n]$ converts the $\mathcal{B}$-coordinates for a vector $\vec{x} \in \mathbb{R}^n$ into the standard coordinates & vice versa:

$$\vec{x} = A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \iff [\vec{x}_{\mathcal{B}}] = A_{\mathcal{B}}^{-1}\vec{x}.$$

$A_{\mathcal{B}}$ is called the change of coordinates matrix from $\mathcal{B}$ to std. basis in $\mathbb{R}^n$.



n-dim V.S. V (imagine $\mathbb{R}^n$ w/ std coords) $\qquad A_{\mathcal{B}} \qquad \mathbb{R}^n$ w/ basis $\mathcal{B}$

Ex:

Convert: ① From $\mathcal{B} = \left\{ \overset{\vec{b}_1}{\begin{pmatrix} 3 \\ -5 \end{pmatrix}}, \overset{\vec{b}_2}{\begin{pmatrix} -4 \\ 6 \end{pmatrix}} \right\}$ into $\mathbb{R}^2$ w/ std coords for

$$[\vec{x}]_{\mathcal{B}} = \begin{array}{cc} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{matrix} c_1 \\ c_2 \end{matrix} \end{array}$$

Ans

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 \Rightarrow \vec{x} = 5\begin{pmatrix} 3 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 15-12 \\ -25+18 \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

② From $\mathbb{R}^2$ w/ std coords into $\mathcal{B} = \left\{ \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{b_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{b_2} \right\}$ for $\vec{x} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Ans: Want c_1, c_2 s.t.

$$\vec{x} = c_1 b_1 + c_2 b_2.$$

$$\Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Hint: $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-4 \\ -3+8 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$