§ 2.8 - 2.9 — Column Space, Null space, Rank, and Nullity

We've used a number of terms so far that hadn't previously been defined. First, we give those formal defs:

Def:

① A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ s.t.

(a) $\vec{0} \in H$;

(b) for all $\vec{u}, \vec{v} \in H$, $\vec{u}+\vec{v} \in H$; and

(c) for all $\vec{v} \in H$, $c\vec{v} \in H$ for all scalars $c \in \mathbb{R}$.

Ex: Which is a ~~vector~~ subspace of the indicated space?

(i) A line through the origin in $\mathbb{R}^2$? → $\boxed{\text{yes!}}$ Let $y = mx$. Then (a) $(0,0)$ on line, (b) if $y_1 = mx_1$ & $y_2 = mx_2 \in H$, then $y_1 + y_2 = mx_1 + mx_2 = m(x_1+x_2) \in H$, and (c) if $y = mx \in H$, then $cy = c(mx) = m(cx) \in H$.

(ii) A line through the origin in $\mathbb{R}^3$?

↳ $\boxed{\text{yes}}$ (see above)

(iii) A line not through the origin?

↳ No. If $y = mx+b$, then $(0,0)$ not in H: $m(0)+b = b$.

(iv) A plane through the origin in $\mathbb{R}^3$?

↳ $\boxed{\text{yes}}$

(v) The unit circle $\{(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi\}$ in $\mathbb{R}^2$?

↳ $\boxed{\text{No}}$ This satisfies none of the criteria.

(vi) Span $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ for any $\vec{v}_1, \dots, \vec{v}_p$. → $\boxed{\text{yes}}$

(vii) $\{\vec{0}\}$ in $\mathbb{R}^n$ for any n. → $\boxed{\text{yes}}$

Def (Cont'd)

② A **basis** for a subspace H in $\mathbb{R}^n$ is a linearly independent subset which ~~satisfies~~ spans H.

Ex: (i) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

↳ • clearly L.I. (not scalar multiples of each other)

• span $= \mathbb{R}^2$?

↳ let $\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$ be any vector. Is $\begin{pmatrix}x\\y\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$?

Can I find c_1, c_2 s.t.

$$(*)\quad \begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} \ ?$$

How about $(*) \iff \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$

$\iff \begin{matrix} x = c_1 \\ y = c_2 \end{matrix}$!

(ii) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\}$ is, too!

↳ • L.I. as above.

• $\begin{pmatrix}x\\y\end{pmatrix} = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\2\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}c_1\\2c_2\end{pmatrix} \Rightarrow \begin{matrix} c_1 = x \\ c_2 = \frac{y}{2} \end{matrix}$.

(iii) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\b\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}b\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}b_1\\0\end{pmatrix}, \begin{pmatrix}0\\b_2\end{pmatrix}\right\}$ too!

(for $b, b_1, b_2 \neq 0$!)

(iv) $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$? → yes!

(v) $\left\{\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}\right\}$ basis for $\mathbb{R}^3$? → No!

(vi) $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right\}$ <u>isn't</u> a basis of $\mathbb{R}^2$.

(vii) $\left\{ \begin{pmatrix}1\\0\end{pmatrix} \right\}$ <u>isn't</u> a basis of $\mathbb{R}^2$.

notation: $\dim(H)$

<u>Def (Cont'd)</u>

③ The <u>dimension</u> of a nonzero subspace H in $\mathbb{R}^n$ is the number of vectors in any basis for H.

(i) The dimension of $\mathbb{R}^n$ is n. (e.g. $\dim(\mathbb{R}^2) = 2$, $\dim(\mathbb{R}^3) = 3$)

(ii) The dimension of a line through the origin $= 1$.

↳ Such a line is of form span$\{\vec{v}\}$ for some $\vec{v}$.

(iii) dim(plane thru origin) $= 2$

(iv) $\dim(\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}) =$ # of L.I. vectors in $\{\vec{v}_1, \ldots, \vec{v}_p\}$.

Now, we're going to relate these notions to some <u>new</u> subspaces we're going to define!

Def: The <u>column space</u> of a matrix A is set $\mathrm{Col}(A)$ of all linear combos of columns of A.

↳ • So, $\mathrm{Col}(A) = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_n)$ if $A = (\vec{v}_1 | \cdots | \vec{v}_n)$.

• By previous examples, $\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^m$ (if $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$).

↑ i.e. if $A = m \times n$.

Def: $\mathrm{rank}(A) \stackrel{\text{def}}{=} \dim(\mathrm{Col}(A))$.

Ex: let
$$A = \begin{array}{c} \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix} \end{array}.$$

(a) Is $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix} \in \mathrm{Col}(A)$? $(A|b) \rightarrow \begin{pmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(b) Find a basis for $\mathrm{Col}(A)$.

$A \underset{\text{r.e.}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $\Rightarrow$ $\vec{v}_3 = 5\vec{v}_1 + 3\vec{v}_2$

$\Rightarrow$ $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathrm{span}\{\vec{v}_1, \vec{v}_2\}$

$\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathrm{Col}(A)$.

(c) Find $\mathrm{rank}(A)$.

$\mathrm{rank}(A) = 2$.

Def: The <u>nullspace</u> of a matrix A is the set $\mathrm{Nul}(A)$ of all solutions to the eq $A\vec{x} = \vec{0}$.

Ex. $A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix} \Rightarrow A\vec{x} = \vec{0} \leftrightarrow (A \,\vdots\, \vec{0})$

$$= \left(\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ -4 & 6 & -2 & 0 \\ -3 & 7 & 6 & 0 \end{array}\right).$$

This matrix is r.e. to

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{(by previous example)},$$

so

$$\begin{aligned} x_1 \quad + 5x_3 &= 0 \\ x_2 + 3x_3 &= 0 \end{aligned} \quad \leftrightarrow \quad \begin{aligned} x_1 &= -5x_3 \\ x_2 &= -3x_3 \\ x_3 &= \text{free} \end{aligned} \quad \longleftrightarrow \quad x_3 \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}$$

Hence, $\mathrm{Nul}(A) = \left\{ x_3 \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} : x_3 \in \mathbb{R} \right\}$

$= \operatorname{span}\{\vec{v}\}$, where $\vec{v} = \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}$.

Fact: If $A = m \times n$, then $\mathrm{Nul}(A) =$ a subspace of $\mathbb{R}^n$.

Def: The Nullity of A is $\dim(\mathrm{Nul}(A))$.

Ex: For A as above, $\mathrm{Nul}(A) = \operatorname{span}\{\vec{v}\}$ for $\vec{v} = \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}$

so nullity of $A = \dim(\mathrm{Nul}(A)) = 1$, as $\vec{v}$ is a basis for $\mathrm{Nul}(A)$.

To summarize: If $A = m \times n$,

space	col (A)	nul (A)
subspace of	$\mathbb{R}^m$	$\mathbb{R}^n$
its dimension is called	"rank"	"nullity"
in terms of linear transforms $T(\vec{x}) = A\vec{x}$	range (T)	kernel (T)

see below

Ex: let $T(\vec{x}) = \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}}_{A = 3\times 4} \vec{x}$ from $\mathbb{R}^4 \to \mathbb{R}^3$.

• Note 1: A r.e. $\begin{array}{cccc} \begin{pmatrix} 1 & 0 & 0 & -1/9 \\ 0 & 1 & 0 & 14/9 \\ 0 & 0 & 1 & 1/3 \end{pmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_4 \end{array}$ $\Rightarrow \vec{v}_4 = -1/9\,\vec{v}_1 + 14/9\,\vec{v}_2 + 1/3\,\vec{v}_3$.

$\Rightarrow$ span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ = span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

subspace of $\mathbb{R}^3$

So: • col(A) = span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ $\Rightarrow$ range (T) = span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

• rank(A) = dim(col(A)) = 3

• Note 2: $A\vec{x} = \vec{0} \leftrightarrow (A|\vec{0})$ r.e. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/9 & 0 \\ 0 & 1 & 0 & 14/9 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \end{array}\right)$

$\leftrightarrow$ $x_1 = 1/9\,x_4$ $\quad x_2 = -14/9\,x_4$ $\quad x_3 = -1/3\,x_4$ $= x_4 \begin{pmatrix} 1/9 \\ -14/9 \\ -1/3 \\ 1 \end{pmatrix}$

• Nul(A) = span $\{\vec{v}\}$ where $\vec{v} = \begin{pmatrix} 1/9 \\ -14/9 \\ -1/3 \\ 1 \end{pmatrix}$ ← subspace of $\mathbb{R}^4$

$\Rightarrow$ kernel (T) = span$(\vec{v})$.

• Nullity = dim(Nul(A)) = 1.

Rank + Nullity Thm

If $A = m \times n$, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

$$\Leftrightarrow \quad \dim(\text{col}(A)) + \dim(\text{nul}(A)) = n.$$

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other important matrix subspaces

Def: The row space of A is the set Row(A) of all linear combinations of the rows of A. $\Leftrightarrow$ $\text{row}(A) = \text{col}(A^T)$, $\text{row}(A^T) = \text{col}(A)$

Ex:

$$A = \begin{array}{c} \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix} \end{array} \begin{array}{l}]\vec{v}_1 \\ ]\vec{v}_2 \\ ]\vec{v}_3 \end{array} \Rightarrow \underbrace{\text{row}(A) = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}}_{\text{Need to (possibly) simplify}}$$

Fact: If A r.e. B, then $\text{row}(A) = \text{row}(B)$.] NOT TRUE FOR COL SPACE!!

$\hookrightarrow$ A r.e. $\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{row}(A) = \text{span}\left\{\begin{pmatrix}1\\0\\5\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

$$= \underbrace{\text{span}\left\{\begin{pmatrix}1\\0\\5\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}\right\}}_{\text{subspace of } \mathbb{R}^3.}$$

Observe: $\dim(\text{row}(A)) = 2$ & $\dim(\text{col}(A)) = 2$ (from before)

Theorem: $\text{Rank}(A) = \dim(\text{row}(A)) = \dim(\text{col}(A))$

$$= \dim(\text{col}(A^T)) = \dim(\text{row}(A^T))$$