§ 2.2 – Inverse of a matrix

- we can't divide matrices, but sometimes, we can find a "multiplicative inverse."

Ex: If $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Def: An $n \times n$ matrix A is invertible if such a matrix exists, i.e. if there exists a matrix B s.t. $AB = I_n = BA$.

- invertible aka non-singular
- not invertible aka singular.

- If A has an inverse, we denote it as A^{-1}!

Q: When does A^{-1} exist?

Ans: When $\det(A) \neq 0$!

Ex: $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ has no inverse!

Q: If A has an inverse, how do we find it?

Ex: Find inverse of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

↳ ① Form $[A \mid I_n]$: $\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$

② Put left part (the A part) into RREF (it equals I_n)

$$\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_2 = R_2 - 2R_1]{}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) \xrightarrow[R_1 = R_1 + R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -2 \end{array}\right)$$

$$\xrightarrow[R_2 = -1 \cdot R_2]{} \Big(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{\substack{\text{RREF} \\ (= I_2)}} \Big| \underbrace{\begin{matrix} 1 & -1 \\ -1 & 2 \end{matrix}}_{A^{-1}} \Big)$$

③ The new "right part" is A^{-1}.

↳ $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Ex: Find inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \boxed{10} \end{pmatrix}$

det $\neq 0$ if this entry $\neq 9$. We did this before w/o knowing it.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 = R_3 - 7R_1]{R_2 = R_2 - 4R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -11 & -7 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + 6R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 6 & -3 \\ 0 & -3 & 0 & 2 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 + \frac{2}{3}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -4/3 & 1 \\ 0 & \boxed{-3} & 0 & 2 & -11 & 6 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 = -1/3 R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -4/3 & 1 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

The inverse!!

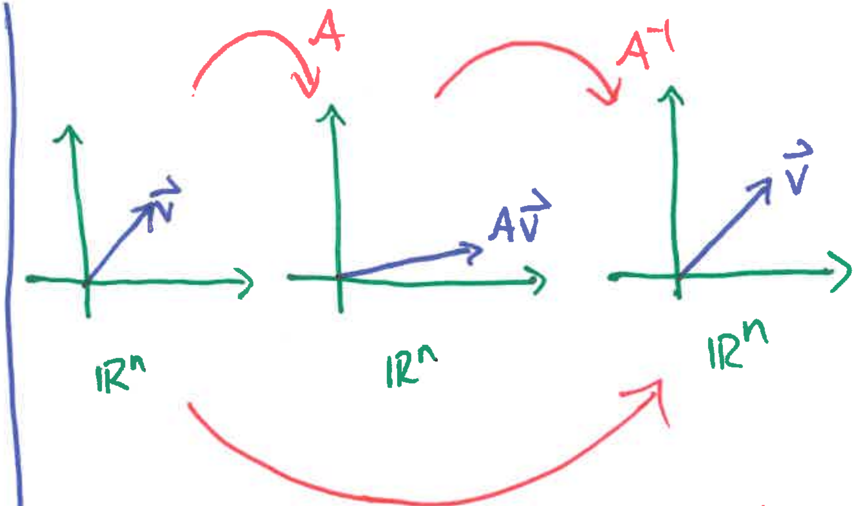
Ans: $\begin{pmatrix} -2/3 & -4/3 & 1 \\ -2/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

Properties of inverses

① $(A^{-1})^{-1} = A$

② $(AB)^{-1} = B^{-1}A^{-1}$

③ $(A^T)^{-1} = (A^{-1})^T$



So, doing A^{-1} after A (or vice versa) gives the original vector.

Ex: Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$.

(a) Find A^{-1} & B^{-1}.

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$$

(b) Find AB.

$$AB = \begin{pmatrix} 7 & 3 \\ 4 & 2 \end{pmatrix}$$

(c) Find $(AB)^{-1}$

$$(AB)^{-1} = \begin{pmatrix} 1 & -3/2 \\ -2 & 7/2 \end{pmatrix}$$

Not Equal!

(d) Find $A^{-1}B^{-1}$.

$$A^{-1}B^{-1} = \begin{pmatrix} 1 & -2 \\ -3/2 & 7/2 \end{pmatrix}$$

Equal!

So:

$(AB)^{-1} = B^{-1}A^{-1}$
$\neq A^{-1}B^{-1}$
in general.

(e) Find $B^{-1}A^{-1}$.

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & -3/2 \\ -2 & 7/2 \end{pmatrix}$$

How to use A^{-1}? $\boxed{\text{To SOLVE } A\vec{x}=\vec{b}!}$

Ex: Solve

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 + x_2 &= 1 \end{aligned} \longleftrightarrow \quad A\vec{x}=\vec{b} \text{ where}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Could solve the old way: $\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow[\text{REF}]{} \cdots$

$$\Rightarrow \begin{aligned} x_1 &= 2 \\ x_2 &= -1 \end{aligned}$$

New way:

$$A\vec{x}=\vec{b} \iff \vec{x} = A^{-1}\vec{b}!$$

↳ From before, $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$$\Rightarrow \vec{x} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix}$$

Ex: Solve

$$x_1 + 2x_2 + 3x_3 = 1$$
$$4x_1 + 5x_2 + 6x_3 = -2$$
$$7x_1 + 8x_2 + 10x_3 = 4$$

or state that no solution exists.

Ans: This is $A\vec{x} = \vec{b}$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$.

Using inverse stuff, we know that if A^{-1} exists, then

$$A\vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b}.$$

From a previous example, A^{-1} does exist and

$$A^{-1} = \begin{pmatrix} -2/3 & -4/3 & 1 \\ -2/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{pmatrix},$$

So $\vec{x} = A^{-1}\begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2/3 + 8/3 + 4 \\ -2/3 - 22/3 - 8 \\ 1 + 4 + 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -16 \\ 9 \end{pmatrix}$. ■

• Also: A^{-1} tells us a lot about a matrix / system / transformation!

↳ See Handouts.