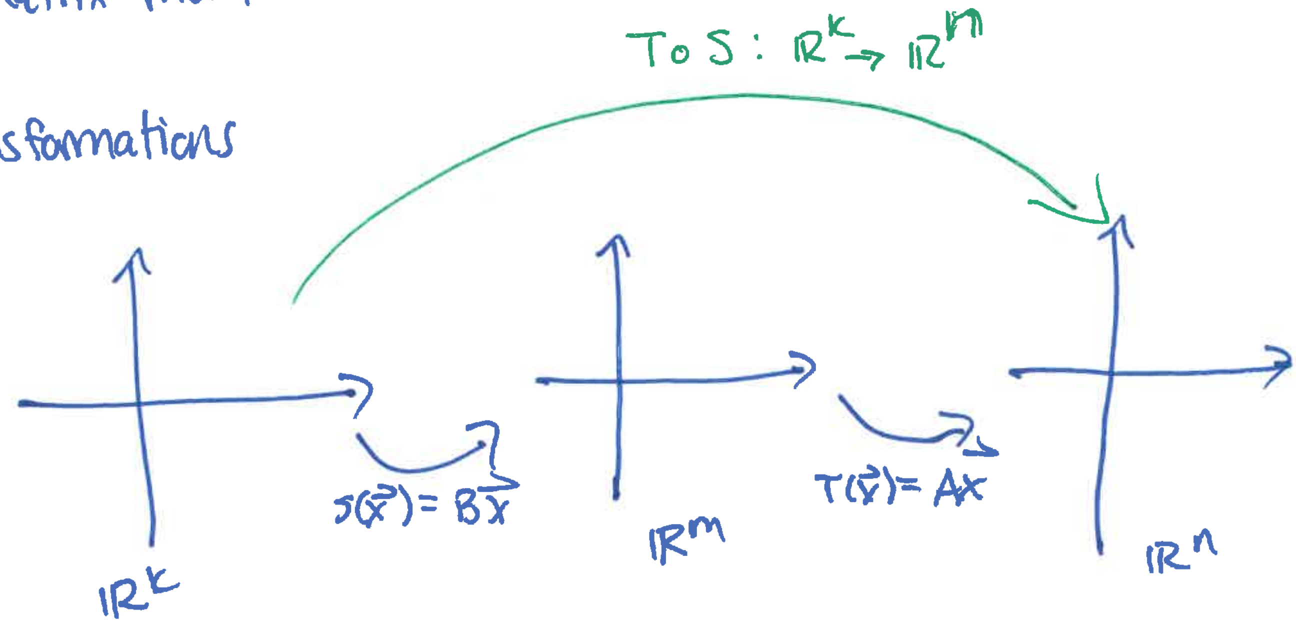
§ 2.1 – Matrix operations + Determinants ← I'm not following the book here.

Recall: Matrix mult.

- As transformations



Given $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$ $\quad (\Rightarrow A = n \times m)$

$S : \mathbb{R}^k \to \mathbb{R}^m$, $S(\vec{x}) = B\vec{x}$, $\quad (\Rightarrow B = m \times k)$

then the composition transformation $T \circ S : \mathbb{R}^k \to \mathbb{R}^n$ is

given by
$$(T \circ S)(\vec{x}) = (AB)\vec{x}$$

matrix multiplication!

$(AB) = (n \times m)(m \times k) = n \times k$!

Recall: ① $AB \neq BA$ in general.

② $AB = AC \not\Rightarrow B = C$ in general.

③ $AB = 0 \not\Rightarrow A = 0$ or $B = 0$ in general.

0-matrix

Def: If A is a square matrix, ← $n \times n$, say

$$A^k = \underbrace{A \cdot A \cdot \cdots A}_{k\text{-times}}$$ ← matrix mult.

Note: $A^0 \stackrel{\text{def}}{=} I_n = n \times n$ identity

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Transpose

The **transpose** A^T ~~is~~ of an $m \times n$ matrix A is the $n \times m$ matrix ~~___~~ whose columns are the rows of A.

Ex: ① $\begin{array}{c} \text{①} \\ \text{②} \end{array}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{array}{cc} \text{①} & \text{②} \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}} \end{array}$

② $\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^T}_{2\times 3} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{3\times 2}$

Properties:
① $(A^T)^T = A$
② $(A+B)^T = A^T + B^T$
③ $(rA)^T = rA^T$, $r \in \mathbb{R}$ scalar
④ $(AB)^T = B^T A^T$ ← not a typo!

Determinants:

Recall: ① If $T(\vec{x}) = A\vec{x}$ is linear, obv trans. from $\mathbb{R}^n \to \mathbb{R}^n$, then $A =$ nxn matrix & ② T maps "parallelograms" to "parallelograms."

③ The vectors $\vec{e}_1, \dots, \vec{e}_n$ (cols of I_n = nxn identity) form the "unit cube" in $\mathbb{R}^n$.

Q: What is the "volume" of the "parallelogram" ~~[illegible]~~ in terms of the unit cube?
↑ spanned by $T(\vec{e}_1), \dots, T(\vec{e}_n)$

Ans: Determinants! (only valid for square matrices!!)

- The goal will be to write

$$\left(\text{hypervolume of } T(\text{parallelogram})\right) = \# \cdot \left(\text{hypervolume of parallelogram}\right)$$

but to find "#", we need to compute determinants.

The method we'll talk about in this class is called cofactor expansion.

Fact: The determinant of a 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $\det(A) = ad - bc$

Ex:

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 1$$

$$\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2$$

- For square matrices larger than 2×2, we do cofactor expansion along the first row.

Ex:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \boxed{+}(1)\det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$$

matrix remaining after deleting 1's row & col

$$\boxed{-}(2)\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix}$$

matrix remaining after deleting 2's row & col

$$\boxed{+}(3)\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

matrix remaining after deleting 3's row & col

Fill in last using

Alternating "+" and "−"

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (45-48) - 2(36-42) + 3(32-35)$$
$$= (-3) - 2(-6) + 3(-3)$$
$$= -3 + 12 - 9 = \boxed{0}$$

- Can use <u>any</u> row/column as long as we get our coefficients correct:

Ex:

$$\det\begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{-}(0)\det\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \boxed{+}(1)\det\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \boxed{-}(0)\det\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

$$= 0 + 1 - 0 = 1$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Ex: $\det \underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 2 & 5 & 1 & 2 \end{pmatrix}}_{A}$

4x4 Signs

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Hint: Pick row/col w/ most zeros! (col 1 or row 3 here)

~~Defn~~ By def,

$$\det(A) = +1\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 5 & 1 & 2 \end{pmatrix} - \cancelto{0}{0} + \cancelto{0}{0} - 2\det\begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left(-\cancelto{0}{0} + 1\det\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}\right)$$

$$-2\left(+\cancelto{0}{0} - 1\det\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} + 1\det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\right)$$

Signs 3x3

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

signs

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (2-15) - (1-10) - 2\left(-(6-4) + (4-3)\right)$$

$$= -13 + 9 - 2(-2+1)$$

$$= -13 + 9 - 2(-1)$$

$$= -13 + 9 + 2 =$$

$$= \boxed{-2}$$

Interpretation: Given a 4D parallelepiped S in $\mathbb{R}^4$ & $T: \mathbb{R}^4 \to \mathbb{R}^4$ s.t. $T(\vec{x}) = A\vec{x}$, the hypervolume of $T(S)$ is

$$(\text{hypervolume } T(S)) = -2 \cdot \text{hypervolume}(S)$$

volumes can't be negative; this sign means we "flipped orientation."