§ 1.9 - The Matrix of a Linear Transformation

- These days, we're imagining matrix multiplication as a function. However, sometimes we don't know what the matrix is!

Ex: Let $\vec{e}_1$ & $\vec{e}_2$ be the columns of $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ & spse T is a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ s.t.

$$T(\vec{e}_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}.$$

Find a formula for T describing the image of $\vec{x}$ in $\mathbb{R}^2$.

Ans: we're going to use linearity of T.

↳ • write $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= x_1 \vec{e}_1 + x_2 \vec{e}_2.$$

- Apply T: $T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$
- Use linearity: $= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$
- Use given: & simplify $= x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{pmatrix}.$$

Note: From here, we know

$$x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) = [T(\vec{e}_1) \ \ T(\vec{e}_2)] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow T(\vec{x}) = A\vec{x}$ where $A = [T(\vec{e}_1) \ \ T(\vec{e}_2)] = \begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix}$!

Theorem:

If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transform, then there exists a unique $n \times m$ matrix

$$A = \left[T(\vec{e}_1) \; \cdots \; T(\vec{e}_m) \right]$$

"standard matrix for a linear transformation".

s.t. $T(\vec{x}) = A\vec{x}$ for all x in $\mathbb{R}^m$. Here, $\vec{e}_j$ is the j^{th} column of the $m \times m$ identity matrix:

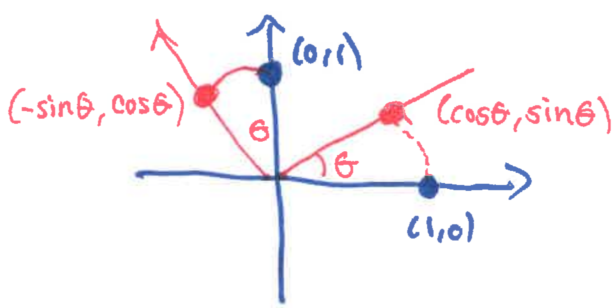
$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots & & \vdots \\ \vdots & \vdots & & 1 & & 0 \\ 0 & 0 & & \vdots & \ddots & \vdots \\ & & & 0 & & 1 \end{pmatrix}}_{m} \Bigg]\, m$$

$\vec{e}_j$ = j^{th} column

= 1 in position j & 0 elsewhere.

Ex: ① $T(\vec{x}) = 3\vec{x}$ in $\mathbb{R}^2$

② $T: \mathbb{R}^2 \to \mathbb{R}^2$ ~~multiply~~ rotates each point about the origin by θ-units ccw.



Recall: T is linear if

$$\text{LHS} \to T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v}) \leftarrow \text{RHS}$$

for all real #'s c, d
vectors $\vec{u}, \vec{v} \in$ domain (T).

Fact: If T linear, then $T(\vec{0}) = \vec{0}$.

Ex: Determine whether each of the following transformations $T: \mathbb{R}^3 \to \mathbb{R}^3$ is linear.

1. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xmapsto{T} \begin{pmatrix} 0 \\ x_3 \\ -x_2 \end{pmatrix}$

If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, then $\boxed{\text{LHS} = \text{RHS}}$!

LHS $= \begin{pmatrix} 0 \\ cu_3 + dv_3 \\ -cu_2 - dv_2 \end{pmatrix}$; RHS $= c\begin{pmatrix} 0 \\ u_3 \\ -u_2 \end{pmatrix} + d\begin{pmatrix} 0 \\ v_3 \\ -v_2 \end{pmatrix}$

2. $\vec{x} \xmapsto{T} 2\vec{x}$ $\boxed{\text{LHS} = \text{RHS}!}$

3. $\vec{x} \longmapsto 2\vec{x} + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$.

$c\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + d\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \left(2c\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + 2d\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \text{LHS}$

But: RHS $= c\left(\begin{pmatrix} 2u_1 \\ 2u_2 \\ 2u_3 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\right) + d\left(\begin{pmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}\right)$

Ex: Find the matrix of 1

Recall: All we need is $T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3)$:

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

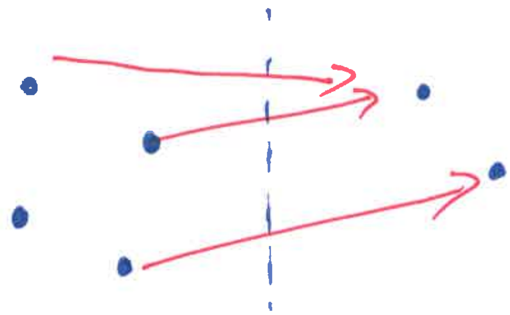
$\Rightarrow A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ works!

Def: $T: \mathbb{R}^m \to \mathbb{R}^n$

- T **onto** if each $\vec{b} \in \mathbb{R}^n$ is the image of **at least** one $x \in \mathbb{R}^m$
- T **one-to-one** if ... **at most** one $x \in \mathbb{R}^m$

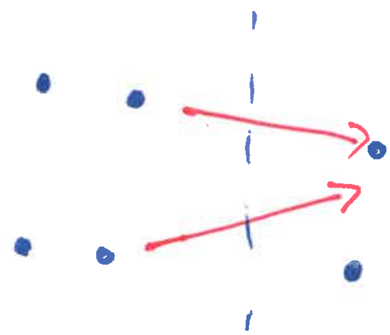
Visual

onto



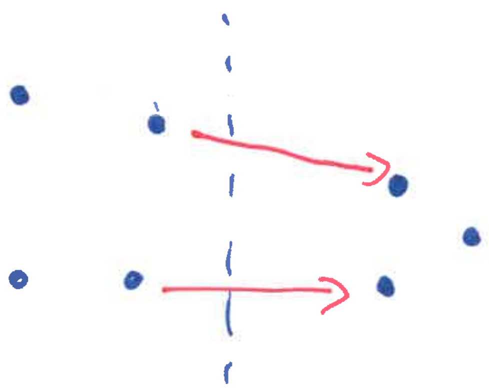
(not one-to-one)

not onto



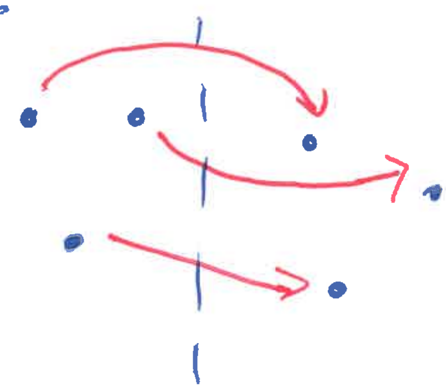
(also not one-to-one)

one-to-one



(not onto)

both



Ex: Let $T: \mathbb{R}^4 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

Is T one-to-one?

↳ **Ans:** yes $\overset{\text{iff}}{\Longleftrightarrow}$ • every $\vec{b} \in \mathbb{R}^3$ gets hit by ≤ 1 vector $\vec{x} \in \mathbb{R}^4$.

do example later

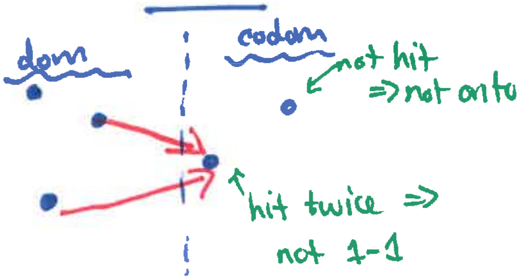
• consider $A\vec{x} = \vec{b}$: Free var $\Rightarrow$ ∞-many solutions!

since $A\vec{x} = b$ has ∞-many sol'ns, T not one-to-one!

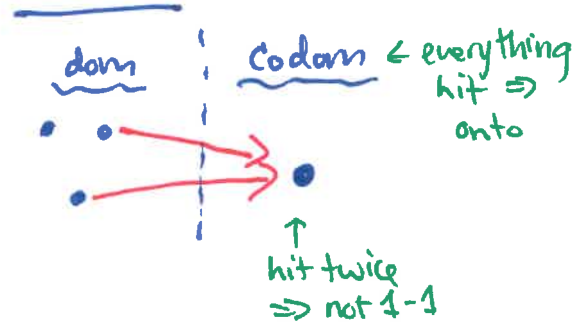
Recall: A linear transformation T is:

- onto / surjective if for all $\vec{b} \in$ codomain (T), there exists an $x \in$ domain (T) w/ $T(\vec{x}) = \vec{b}$
- one-to-one (1-1) / injective if, for any $\vec{b} \in$ codomain(T), there is no more than one vector $x \in$ domain (T) s.t. $T(\vec{x}) = \vec{b}$

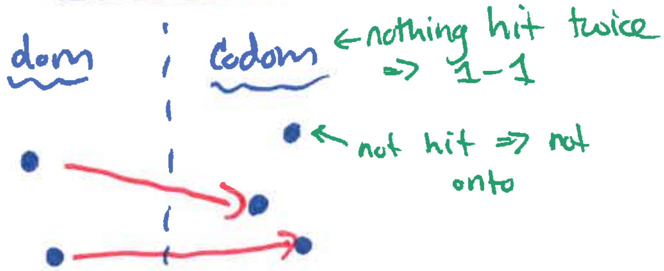
neither



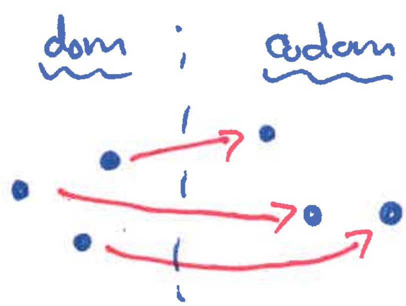
onto
not 1-1



1-1
not onto



1-1
onto



- The goal will be to understand injective / surjective funcs in the context of lin. alg. we already know!

Ex: ① If $T(\vec{x}) = A\vec{x}$ and $A\vec{x} = \vec{b}$ has a free var, the T not one-to-one!

↳ • If $A\vec{x} = \vec{b}$ has a free var, then there are ∞-many $\vec{x}$ s.t. $T(\vec{x}) = \vec{b}$.

• If there are ∞-many $\vec{x}$ s.t. $T(\vec{x}) = \vec{b}$, there aren't ≤ 1 such $\vec{x}$.

Ex: $T: \mathbb{R}^4 \to \mathbb{R}^3$, $T(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}$.

Then: $A\vec{x} = \begin{pmatrix} b_1 \\ \vdots \\ b_3 \end{pmatrix}$ is

$$\begin{aligned} x_1 - 4x_2 + 8x_3 + x_4 &= b_1 \\ 2x_2 - x_3 + 3x_4 &= b_2 \\ 5x_4 &= b_3 \end{aligned}$$

free var!

not 1-1

② If there is a $\vec{b} \in \text{codom}(T)$ where $A\vec{x} = \vec{b}$ has no solution, then $T(\vec{x}) = A\vec{x}$ not onto.

↳ Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. The vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ does not equal $T(\vec{x})$ for any $\vec{x}$, since

$$T(\vec{x}) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{aligned} x &= 1 \\ y &= 1 \\ 0 &= 1. \end{aligned}$$

$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

not pass.!

not onto!

Facts: ① T onto iff $T(\vec{x}) = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$ (let A be s.t. $T(\vec{x}) = A\vec{x}$, $A = (\vec{a}_1 | \cdots | \vec{a}_m)$)

$T: \mathbb{R}^m \to \mathbb{R}^n$

$$\Updownarrow$$

$\boxed{\text{the columns of } A \text{ span } \mathbb{R}^n \text{ (i.e. } \operatorname{span}\{\vec{a}_1, \ldots, \vec{a}_m\} = \mathbb{R}^n)}$

② T 1-1 iff the <u>only</u> vector $\vec{x}$ s.t. $T(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$ (let A be s.t. $T(\vec{x}) = A\vec{x}$)

$$\Updownarrow$$

$\boxed{\text{the columns } \vec{a}_1, \ldots, \vec{a}_m \text{ of } A \text{ are L.I}}$

Theorem.

Ex: Consider $T(x_1, x_2) = \langle 3x_1 + x_2,\ 5x_1 + 7x_2,\ x_1 + 3x_2 \rangle$.

Is T one-to-one? onto?

Note: $T: \mathbb{R}^2 \to \mathbb{R}^3$

↳ <u>Note</u>: T <u>is</u> linear! $T(\vec{x}) = \overbrace{\begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}}^{A} \vec{x}$, $\vec{a}_1$, $\vec{a}_2$

- By theorem, T one-to-one $\Longleftrightarrow$ cols of A are LI. This is true since $\vec{a}_1$ not a scalar multiple of $\vec{a}_2$. **one-to-one!**

- By thm, T onto $\Longleftrightarrow$ span $\{\vec{a}_1, \vec{a}_2\} = \mathbb{R}^3$. But:
  - span of 2 vectors has dim ≤ 2
  - $\mathbb{R}^3$ has dim $= 3$
  - So span $\{\vec{a}_1, \vec{a}_2\} \neq \mathbb{R}^3$.

In terms of pivots:
- A has 2 pivots
- For span{cols A} $= \mathbb{R}^3$, need 3 pivots

Hence, <u>not</u> onto!

$$\text{REF}(A) = \begin{pmatrix} \boxed{1} & 3 \\ 0 & \boxed{-8} \\ 0 & 0 \end{pmatrix}$$