§1.8. Linear Transformations

Before: A = $m \times n$ matrix

$\vec{v}$ = $n \times 1$ vector

$\Downarrow$

$A\vec{v}$ = $m \times 1$ vector

$\updownarrow$

Multiply $\vec{v}$ by A to get $A\vec{v}$

$\longleftrightarrow$

Now: A = function w/ "domain" a set of $n \times 1$ vectors & "codomain" a set of $m \times 1$ vectors.

$\updownarrow$

plug $\vec{v}$ into A to get $A(\vec{v})$.

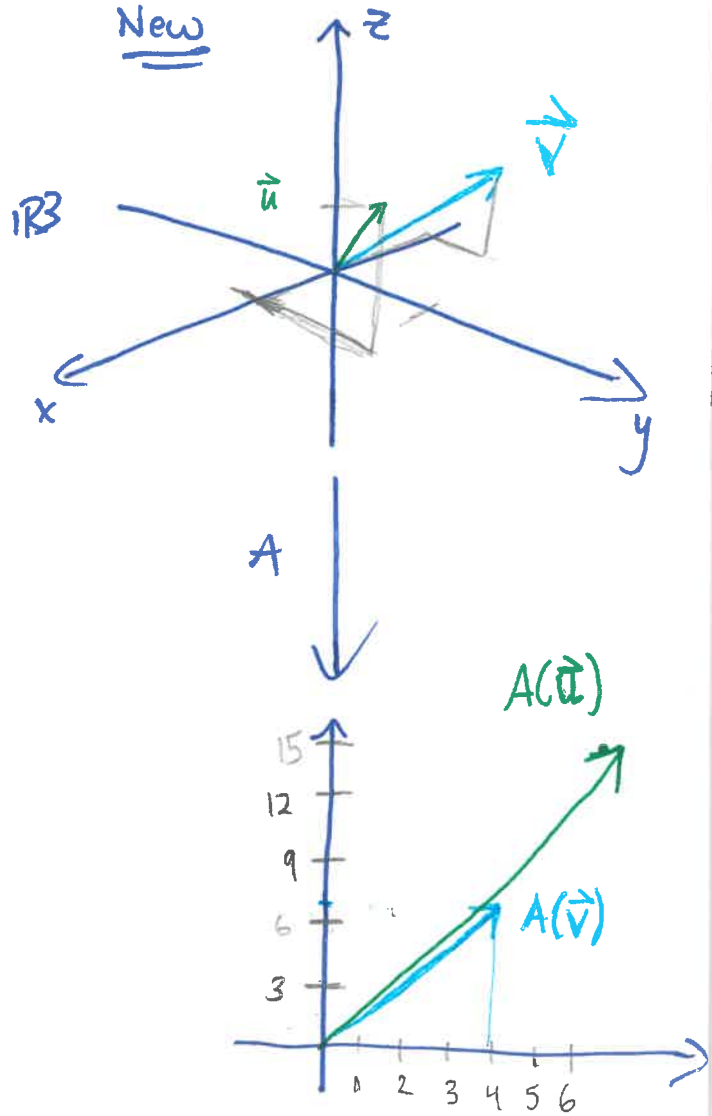
Ex: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Then if $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$,

old

$$A \cdot \vec{u} = \begin{pmatrix} 1+2+3 \\ 4+5+6 \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \end{pmatrix}$$

$$A \cdot \vec{v} = \begin{pmatrix} -1+2+3 \\ -4+5+6 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

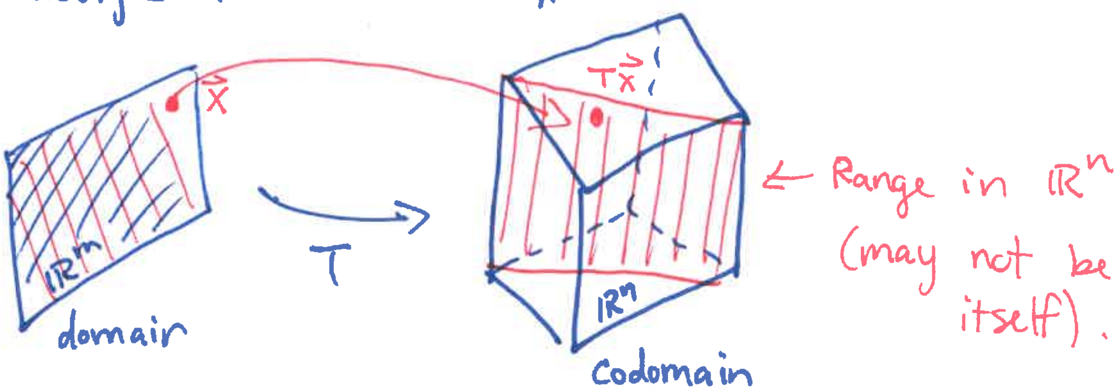
New



↑ ignore this. :P

- In this example, $A = 2\times 3$ & solving the eq. $A\vec{x} = \vec{b}$ corresponds to finding all vectors in $\mathbb{R}^3$ that are transformed into $\vec{b}$ in $\mathbb{R}^2$.

<u>Def</u>: A <u>transformation</u> T from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a function (or rule) that assigns to each vector $\vec{v}$ in $\mathbb{R}^m$ the vector $T\vec{v}$ in $\mathbb{R}^n$.



- $\mathbb{R}^m$ = domain
- $\mathbb{R}^n$ = codomain
- set of all images $T(\vec{x})$ in $\mathbb{R}^n$ = range.

<u>Ex</u>: let $A = \underbrace{\begin{bmatrix} 1 & 3 & -1 \\ -3 & 5 & 7 \end{bmatrix}}_{2\times 3}$, $\vec{u} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$,

and define a transform $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\vec{x}) = A\vec{x}$, i.e.

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 - x_3 \\ -3x_1 + 5x_2 + 7x_3 \end{pmatrix}.$$

(a) Find $T(\vec{u})$ [the image of $\vec{u}$]

(b) Find $\vec{x}$ in $\mathbb{R}^3$ whose image under T is $\vec{b}$.

(c) Is there more than one $\vec{x}$ whose image under T is $\vec{b}$?

(d) Is $\vec{c}$ in Range of T?

Ex (cont'd)

(a) $T(\vec{u}) = A\vec{u} = \begin{pmatrix} 1 & 3 & -1 \\ -3 & 5 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 2-3+0 \\ -6-5+0 \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \end{pmatrix}$$

(b) Want $\vec{x}$ such that $T(\vec{x}) = \vec{b} \iff A\vec{x} = \vec{b}$

$\iff \vec{x}$ solves system w/ augmented matrix $[A \mid \vec{b}]$

So $\left(\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ -3 & 5 & 7 & 2 \end{array}\right) \xrightarrow{R_2 = R_2 + 3R_1} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & 14 & 4 & 11 \end{array}\right)$

$x_1 + 3x_2 - x_3 = 3$

① $x_1 = 3 + x_3 - 3x_2$

$\Rightarrow x_1 = 3 + x_3 - \frac{3}{14}(11 - 4x_3)$

② $14x_2 + 4x_3 = 11$

$x_2 = \frac{1}{14}(11 - 4x_3)$

Hence, any vector of form $(\star) = \begin{pmatrix} 3 + x_3 - 3/14\,(11 - 4x_3) \\ \frac{1}{14}(11 - 4x_3) \\ x_3 \end{pmatrix}$ satisfies

$T(\vec{x}) = \vec{b}$. Ex: $x_3 = 0 \Rightarrow \begin{pmatrix} 9/14 \\ 11/14 \\ 0 \end{pmatrix} \leftarrow 9/14$

(c) yes. By $(\star)$, there are infinitely many such $\vec{x}$!

(d) $\vec{c}$ in Range$(T) \iff T(\vec{x}) = \vec{c}$ some $\vec{x} \iff A\vec{x} = \vec{c}$

consistent $\iff \underbrace{\left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ -3 & 5 & 7 & -5 \end{array}\right)}$ yields consistent system. But

$\xrightarrow{R_2 = R_2 + 3R_1} \left(\begin{array}{ccc|c} 1 & 3 & -1 & 2 \\ 0 & 14 & 4 & 1 \end{array}\right)$ has a solution, so: yes!

3)

Note: (d) not always "yes".

Ex: If $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$ & $\vec{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$, then

$$[A \mid \vec{c}] = \left(\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right) \longrightarrow \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{pmatrix}$$ is not

consistent. Hence, this $\vec{c}$ not in the image of this T!

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ corresponds to projection of $\mathbb{R}^3$ into xy-plane of $\mathbb{R}^2$:

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.$$

Some transformations are particularly important.

Def: Linear transformation is a $\underbrace{\text{map}}_{\text{(or transformation)}}$ $T\underbrace{: \mathbb{R}^n \to \mathbb{R}^m}_{\uparrow \text{ can omit}}$ such that

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

② $T(c\vec{u}) = c \cdot T(\vec{u})$

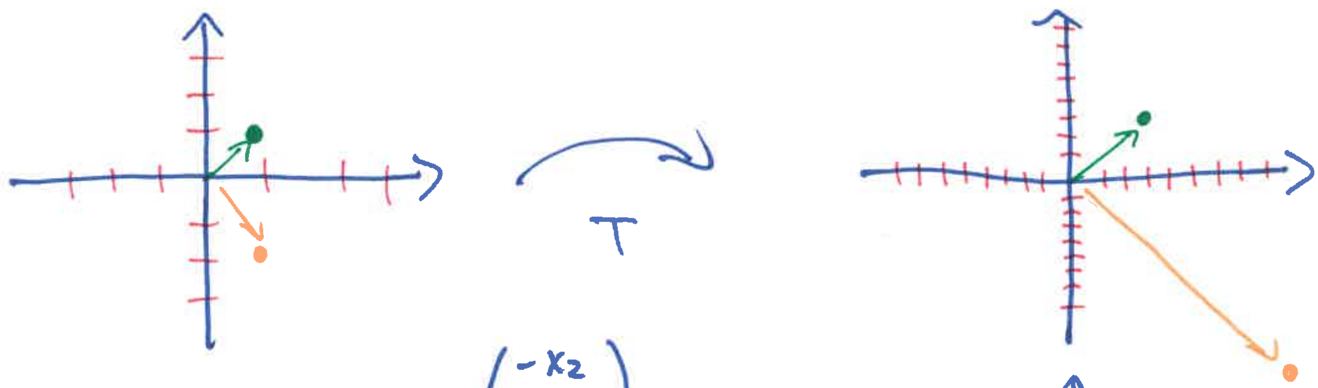
for all scalars c & vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$. use "domain of T"

→ Rewrite as one condition:

③ $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ all scalars c, d
vectors $\vec{u}, \vec{v}$.

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T: \vec{x} \longmapsto 3\vec{x}$ is a linear transform:

$$\begin{aligned} T(c\vec{u}+d\vec{v}) &= 3(c\vec{u}+d\vec{v}) && \text{by def of } T \\ &= 3c\vec{u}+3d\vec{v} && \text{by arith.} \\ &= c(3\vec{u})+d(3\vec{v}) && \text{by rearranging} \\ &= cT(\vec{u})+dT(\vec{v}) && \text{by def of } T. \end{aligned}$$



"same pic" but 3x as big!

Ex: Let $T(\vec{x}) = \overbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{= \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}}$. Note: This is linear.

If $\vec{u} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find $T(\vec{u}+\vec{v})$

Ans: $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ b/c linear

$$= T\begin{pmatrix} 4 \\ 1 \end{pmatrix} + T\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}.$$