§1.7 - Linear Independence

Def: The vector $\vec{0}$ is called <u>the trivial solution</u> to the equation

$$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$$

$$\left(\text{or } \underbrace{[\vec{v}_1 | \cdots | \vec{v}_p]}_{A}\vec{x} = \vec{0},\ \ldots\right).$$

Note: The vector eq.

$$x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$$

<u>always</u> has <u>a</u> solution:

In particular, if $x_1 = 0, \ldots, x_p = 0$, the eq. is true!

Question: when is $\vec{0}$ the <u>only</u> solution?

"Ans": Sometimes but not always.

Ex: ①

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \boxed{x_2 = 0} \ \&$$

$$2x_1 + x_2 = 0 \Rightarrow 2x_1 = 0 \ \boxed{\Rightarrow x_1 = 0.}$$

only trivial sol'n

②

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow 2x_1 + x_2 = 0$$

$$\Leftrightarrow \boxed{\begin{aligned} x_1 &= \frac{x_2}{2} \\ x_2 &= \text{free} \end{aligned}}$$

∞-many sol'n (incl. trivial one!)

Def: The set of vectors $\vec{v}_1, \dots, \vec{v}_p$ is linearly independent if $[\vec{v}_1 | \cdots | \vec{v}_p]\,\vec{x} = \vec{0}$ has only the trivial solution. If not, linearly dependent.

↳ Having nontrivial solution $\iff$ There existing $x_1, \dots, x_p$ not all zero such that

$$x_1 \vec{v}_1 + \dots + x_p \vec{v}_p = \vec{0}$$

Q: If nontrivial solution, how many solutions? $\iff$ ≥ 1 ~~solution~~ of the vectors $\vec{v}_1, \dots, \vec{v}_p$ is a linear combo of the other vectors.

Ans: ∞-many! $\iff A\vec{x} = \vec{0}$ has free vars.

Ex
HW 1 #11(i)

Do $\vec{v}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$ form a linearly independent set?

Ans: Try to solve $\vec{0} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 4 & 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to get free vars!

$$\xrightarrow{R_3 = R_3 - 4R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 2 & 5 \end{pmatrix} \xrightarrow{R_3 = R_3 - \frac{2}{3}R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$-3-4(-2)$ $\qquad$ $5 - \frac{2}{3}(6)$

$\vec{0} = A\vec{x} \iff \vec{0} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \iff$

③ $x_3 = 0$

② $3x_2 + 6(0) = 0 \Rightarrow x_2 = 0$

① $x_1 - 2(0) = 0 \Rightarrow x_1 = 0.$

So: only trivial solution to $A\vec{x} = \vec{0} \Rightarrow$ linearly independent.

Geometrically

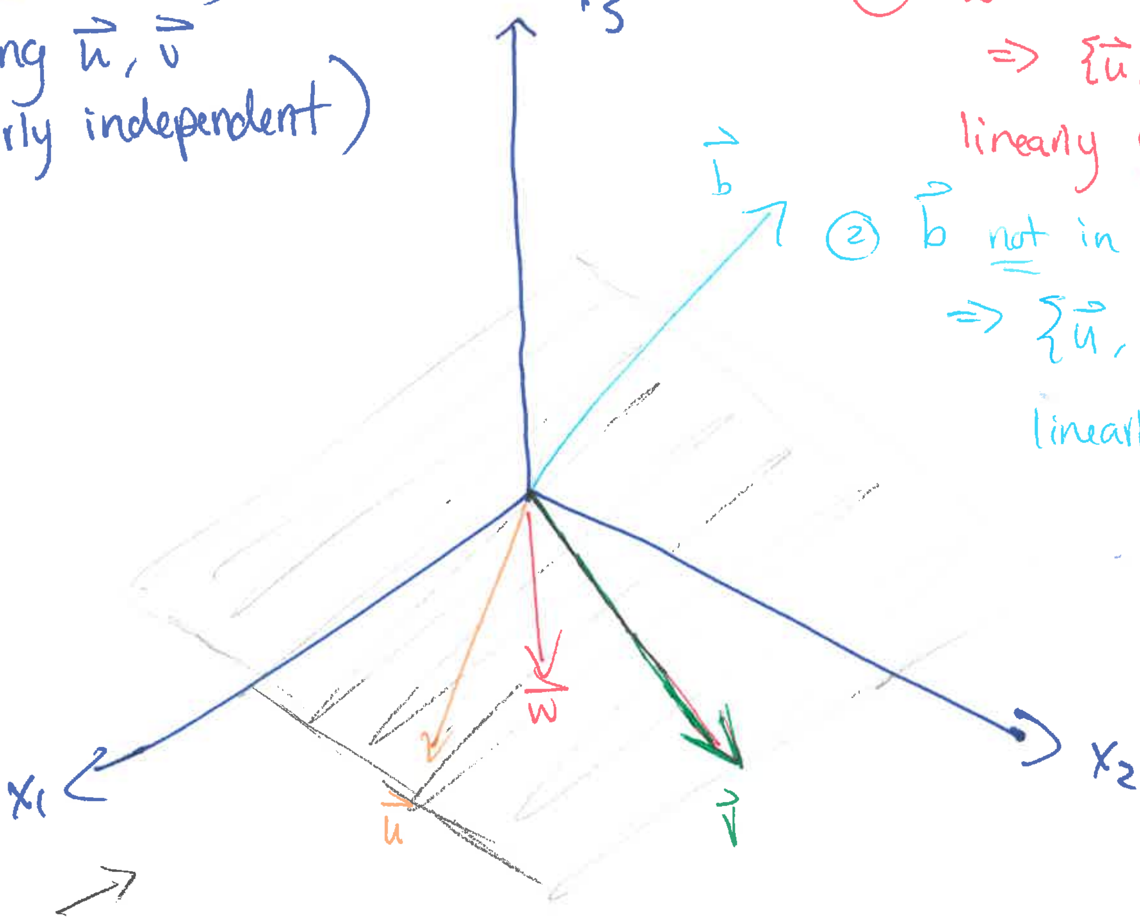
(assuming $\vec{u}, \vec{v}$ linearly independent)

① $\vec{w}$ in span$\{\vec{u}, \vec{v}\}$

$\Rightarrow \{\vec{u}, \vec{v}, \vec{w}\}$ linearly dependent.

② $\vec{b}$ not in span$\{\vec{u}, \vec{v}\}$

$\Rightarrow \{\vec{u}, \vec{v}, \vec{b}\}$ linearly independent.



infinite plane (whole $x_1 x_2$-plane) is span$\{\vec{u}, \vec{v}\}$

§ 1.7 (Cont'd)

Recall: • $\{\vec{v}_1, \dots, \vec{v}_p\}$ is Linearly Independent (L.I.) if the vector equation $x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{0}$ has only the trivial solution and is Linearly dependent (L.D.) otherwise.

↳ $A\vec{x} = \vec{0}$ is called "homogeneous equation" b/c of the "$= \vec{0}$".

From last time: $\{\vec{v}_1, \dots, \vec{v}_p\}$ L.I $\Leftrightarrow$ the system corresponding to augmented matrix $[\vec{v}_1 | \cdots | \vec{v}_p | \vec{0}]$ has no free variables.

Ex: If $\vec{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}4\\5\\6\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}7\\8\\9\end{pmatrix}$, is $\{\vec{v}_1, \dots, \vec{v}_3\}$ L.I.?

$$\begin{pmatrix}1 & 4 & 7 & 0\\ 2 & 5 & 8 & 0\\ 3 & 6 & 9 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 7 & 0\\ 0 & -3 & -6 & 0\\ 0 & -6 & -12 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 7 & 0\\ 0 & -3 & -6 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Not L.I.

HOW TO PICTURE IT GEOMETRICALLY

The vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^m$ is L.I. if

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

is a p-dimensional region in $\mathbb{R}^m$ & L.D. otherwise!

Facts from this

① $\{\vec{v}\}$ is L.I if span $\{\vec{v}\}$ is 1-dim space

$\Updownarrow$

$\vec{v} \neq \vec{0}$. (if $\vec{v} = \vec{0}$, span $= \vec{0} =$ pt (0-dim); otherwise, span = line = 1-dim)

② $\{\vec{u}, \vec{v}\}$ is L.I. if Span $\{\vec{u}, \vec{v}\}$ is 2-dim space (plane)

$\Updownarrow$

~~as~~ $\vec{u}$ & $\vec{v}$ not collinear

$\Updownarrow$

$\vec{u}$ & $\vec{v}$ not scalar multiples of one another.

Ex: (i) $\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}3\\1.5\\0\end{pmatrix}\right\} \rightarrow$ L.D.

$\left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\4\\0\end{pmatrix}\right\} \rightarrow$ L.I.

(ii) Let $\vec{u} = \begin{pmatrix}1\\2\\0\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\1\\0\end{pmatrix}$. Describe span $\{\vec{u}, \vec{v}\}$ & explain why $\vec{w}$ in span$\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ L.D.

③ If a set contains more vectors than there are entries in each vector, then L.D., i.e. $\{\vec{v}_1, \dots, \vec{v}_p\}$ L.D. in $\mathbb{R}^n$ if $p > n$.

Ex: $\left\{\underbrace{\begin{pmatrix}u_1\\u_2\end{pmatrix}}_{\vec{u}}, \underbrace{\begin{pmatrix}v_1\\v_2\end{pmatrix}}_{\vec{v}}, \underbrace{\begin{pmatrix}w_1\\w_2\end{pmatrix}}_{\vec{w}}\right\}$ L.D. b/c L.I $\Leftrightarrow$ span $\{\vec{u}, \vec{v}, \vec{w}\}$ is a 3-dim subspace of $\mathbb{R}^2$, but no such subspaces exist.

(Cont'd)

④ $\underbrace{\{\vec{v}_1, \dots, \vec{v}_p\}}_{=S}$ L.D. if any vector in S is a linear combo of the others.

Why? If this is true, then the system corresponding to $[\vec{v}_1 | \cdots | \vec{v}_p] \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \vec{0}$ will have a free var!

Ex: Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, and $v_3 = v_1 + 2v_2$. Then

$$[\vec{v}_1 | \cdots | \vec{v}_3 | \vec{0}] = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 2 & 0 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & -7 & -14 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & \underset{\uparrow\,1}{\cancel{}} & \underset{\uparrow\,2}{\cancel{}} & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & \underset{\uparrow\,0}{\cancel{}} & 0 \end{pmatrix} \leftarrow \text{Free var!}$$

⑤ If $\{\vec{v}_1, \dots, \vec{v}_p\}$ contains the zero vector, then L.D!

Why? $\vec{0} = 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_{p-1}$, regardless of what $\vec{v}_1, \dots, \vec{v}_{p-1}$ is!

Ex: Do a HW problem!

⑥