§ 1.4 – The matrix equation $A\vec{x} = \vec{b}$

Recall: • Linear combinations, span, how to do matrix multiplication

Now: Every linear combination of vectors can be written as matrix multiplication!

Notation: For vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^m$, the notation

$$\left(\vec{v}_1 \mid \vec{v}_2 \mid \cdots \mid \vec{v}_n \right)$$ ← The book doesn't put the vertical lines!

denotes the $m \times n$ matrix having the vectors $\vec{v}_i$ as columns!

↳ **Ex:** If $\vec{v}_1 = \langle 1, 2 \rangle$ (or $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$), $\vec{v}_2 = \langle 3, 4 \rangle$ (or $= \begin{pmatrix} 3 \\ 4 \end{pmatrix}$), and $\vec{v}_3 = \langle 5, 6 \rangle$ (or $= \begin{pmatrix} 5 \\ 6 \end{pmatrix}$), then

$$[\vec{v}_1 \mid \vec{v}_2 \mid \vec{v}_3] = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$ is 2×3

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we're going to use this notation to write linear combos as matrix multiplication!

Ex: (1) If $\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ & $\vec{v}_2 = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$, then, the linear combo for x_1, x_2 in $\mathbb{R}$,

$$x_1\vec{v}_1 + x_2\vec{v}_2 = x_1\begin{pmatrix} 2 \\ 3 \end{pmatrix} + x_2\begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} -4x_2 \\ 8x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 - 4x_2 \\ 3x_1 + 8x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 \\ 3 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= [\vec{v}_1 \mid \vec{v}_2]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(2) Let $\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$. Write

$3\vec{u}_1 + 7\vec{u}_2 - 11\vec{u}_3$ as a matrix times a vector.

(3) For $\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4$ in $\mathbb{R}^m$, write $3\vec{w}_1 - 5\vec{w}_2 + 7\vec{w}_3 - 9\vec{w}_4$ as a matrix times a vector.

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(2) $[\vec{u}_1 \mid \vec{u}_2 \mid \vec{u}_3]\begin{pmatrix} 3 \\ 7 \\ -11 \end{pmatrix}$

(3) $[\vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3 \mid \vec{w}_4]\begin{pmatrix} 3 \\ -5 \\ 7 \\ -9 \end{pmatrix}$

So: $\vec{b}$ is a linear combo of $\vec{a}_1, \ldots, \vec{a}_n$ $\iff$ $\vec{b} = x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$, some $x_1, \ldots, x_n$ in (by defn)

$\Updownarrow$ (what we just saw)

$$\vec{b} = [\vec{a}_1 | \cdots | \vec{a}_n] \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}}$$

$x_1, \ldots, x_n$ called "the weights"

!!

(see handout for equivalent ways to write this)

Also: The $\underbrace{\text{solution set of the system}}_{\text{(i.e. the weights)}}$ is the same as the solution ~~set~~ vector $\vec{x}$!

§ 1.5: $A\vec{x} = \vec{0}$

Note: $A\vec{x} = \vec{0}$ always has a solution $\vec{x}$, namely $\vec{x} = \vec{0}$:

Ex: $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$.

Then $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0} \Rightarrow \begin{matrix} 2x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix}$ is true if $x_1 = 0$ & $x_2 = 0$

Def: $\vec{x} = \vec{0}$ is the trivial solution to $A\vec{x} = \vec{0}$.

Question : when does $A\vec{x} = \vec{0}$ have a nontrivial solution (i.e. a solution $\vec{x} \neq \vec{0}$).

"Ans": Sometimes but not always!

Ex: ① In the last example,

$$\begin{aligned} &① \; 2x_1 + x_2 = 0 \\ &② \; x_1 + x_2 = 0 \end{aligned} \iff \begin{aligned} &② \; x_1 = -x_2 \\ &① \; 2(-x_2) + x_2 = 0 \\ &\Rightarrow -x_2 = 0 \\ &\Rightarrow \boxed{x_2 = 0} \\ &② \Rightarrow \boxed{x_1 = 0} \end{aligned}$$

② $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0} \xRightarrow{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$

$$\Rightarrow \begin{aligned} x_1 + 2x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= -2x_2 \\ x_2 &= \text{free} \end{aligned}} \leftarrow \infty\text{-many nontrivial solutions!}$$

$A\vec{x} = \vec{0}$

Ans:
- Always have trivial solution
- Linear systems can have 0, 1, or ∞-many solutions.
- So, nontrivial solutions for $A\vec{x} = \vec{0} \iff > 1$ solution $\iff \infty$-many
- ∞-many $\iff$ free var.