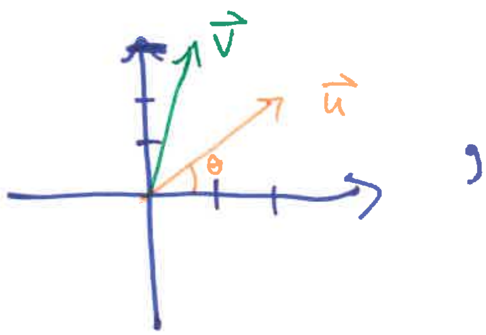
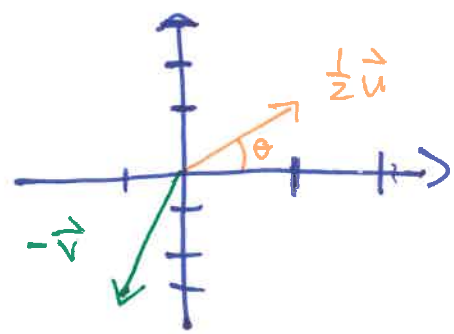
§1.3. – Vector Equations

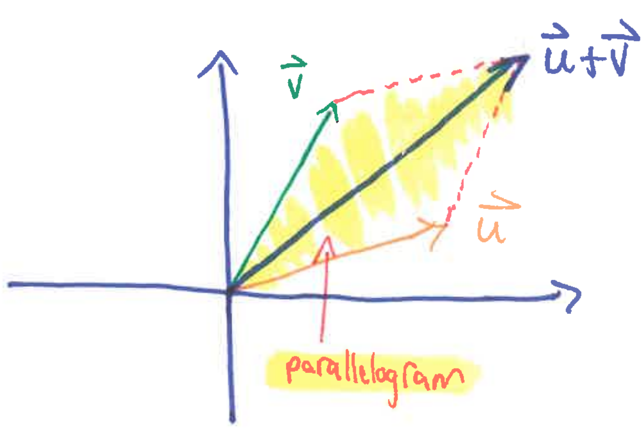
Recall: • In $\mathbb{R}^2$ (or $\mathbb{R}^3$ or $\mathbb{R}^n$ …), a vector can be thought of as an arrow w/ its initial point at the origin.

• vector addition, subtraction, & scalar multiplication can all be visualized graphically:

∘ If $\vec{u}$ & $\vec{v}$ are: 

then here is: $\frac{1}{2}\vec{u}$ & $-\vec{v}$ 

∘ $\vec{u}+\vec{v}$ is the diagonal of the parallelogram formed by $\vec{u}$ & $\vec{v}$:



<u>Note:</u> $\vec{u}-\vec{v}$ is just $\vec{u} + -\vec{v}$!

• These ideas will be useful moving forward!

Def: Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ in $\mathbb{R}^n$ & scalars $c_1, c_2, \dots, c_p$ in $\mathbb{R}$, the vector

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$$

is called the linear combination of $\vec{v}_1, \dots, \vec{v}_p$.

↳ $c_1, \dots, c_p$ are the weights.

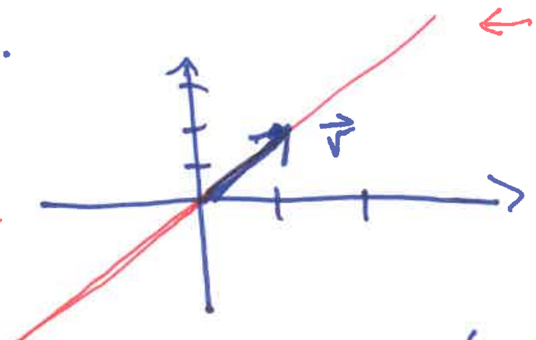
Ex: linear combos of $\vec{v}_1$ & $\vec{v}_2$ include

- $\vec{v}_1$ $(1\vec{v}_1 + 0\vec{v}_2)$, $2\vec{v}_1$ $(2\vec{v}_1 + 0\vec{v}_2)$, $\frac{1}{2}\vec{v}_1$, etc.
- $\vec{v}_2$ $(0\vec{v}_1 + 1\vec{v}_2)$, $-3\vec{v}_2$ $(0\vec{v}_1 - 3\vec{v}_2)$, ..., etc.
- $\vec{v}_1 + \vec{v}_2$, $\vec{v}_1 - \vec{v}_2$, $\sqrt{3}\vec{v}_1 + 16\pi\vec{v}_2$, ..., etc.
- $\vec{0}$ $(= 0\vec{v}_1 + 0\vec{v}_2)$.

Ex: ① Draw the set of all linear combinations of $\vec{v} = \langle 1,2 \rangle$ in $\mathbb{R}^2$.

← Ans: The set of all scalar multiples of $\vec{v}$ is the line in $\mathbb{R}^2$ containing $\vec{v}$.

Note: Linear combos of 1 vector are just scalar multiples



② Let $\vec{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\vec{a}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$. Can $\vec{b}$ be written as a linear combo of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$?

Compare w/ HW1 #7, #8, #9

(Cont'd)

- $\vec{b}$ can be written as a linear combo of $\vec{a}_1, \vec{a}_2,$ and $\vec{a}_3$ iff

$$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 \quad \text{for real numbers } x_1, x_2, x_3$$

$$\Updownarrow$$

$$\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \vec{b} = x_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + x_3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \text{for some } x_1, x_2, x_3$$

$$\Updownarrow$$

$$\begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = \vec{b} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{some } x_1, x_2, x_3.$$

- But this is a linear system! So $\vec{b}$ is a linear combo of $\vec{a}_1, \vec{a}_2, \vec{a}_3$ iff the corresponding linear system has a solution $\langle x_1, x_2, x_3 \rangle$!

↳ Now: Can we solve?

- Augmented matrix $= \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 6 \end{pmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & \boxed{1} & -2 & -3 \\ 0 & -1 & 1 & 6 \end{pmatrix}$

$$\xrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & \boxed{1} & -2 & -3 \\ 0 & 0 & \boxed{-1} & 3 \end{pmatrix}$$

Find x_1, x_2, x_3?

- REF! Because no row $[0, \ldots, 0, b]$, $b \neq 0$, this has a solution $\Rightarrow \vec{b}$ is a linear combo of $\vec{a}_1, \vec{a}_2, \vec{a}_3$!

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Conclusion

A vector $\vec{b}$ can be generated by a linear combo of vectors $\vec{a}_1, \dots, \vec{a}_n$ iff the linear system w/ augmented matrix

$$[\vec{a}_1 \ \ \vec{a}_2 \ \cdots \ \vec{a}_n \ \ \vec{b}]$$

has a solution (or ∞-many!)

Spans

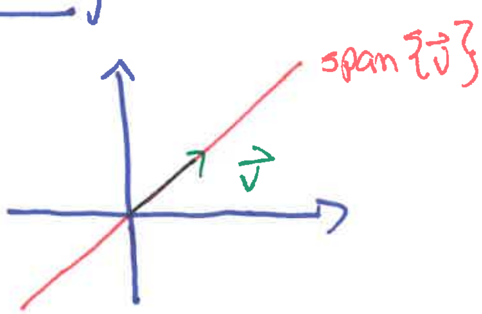
Def: Let $\vec{v}_1, \dots, \vec{v}_p$ be vectors in $\mathbb{R}^n$. Then

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \text{all linear combos of} \\ \vec{v}_1, \dots, \vec{v}_p \end{array} \right\}$$

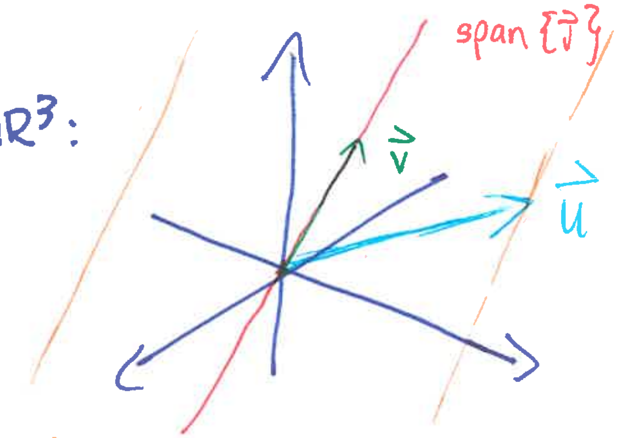
$$= \left\{ \vec{u} : \vec{u} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p \text{ for real \#'s } c_1, \dots, c_p \right\}$$

Geometrically:

$\mathbb{R}^2$:



$\mathbb{R}^3$:



(infinite plane) = span$\{\vec{u}, \vec{v}\}$

Synonyms

$\vec{b}$ is in span $\{\vec{a}_1, \dots, \vec{a}_n\}$ means same as

① b is a linear combo of $\vec{a}_1, \dots, \vec{a}_n$

② $b = c_1\vec{a}_1 + \dots + c_n\vec{a}_n$ for some real #s $c_1, \dots, c_n$

③ The system w/ augmented matrix $[\vec{a}_1 \dots \vec{a}_n \ \vec{b}]$ has some solution

Use for HW1 #11!

$+ \dots$

To be continued!

Ex: $\vec{b}$ is in the span $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ (where all vectors are as in previous example).