§ 1.2 (Cont'd)

Now that we know how to put a matrix into RREF, we'll state some facts + learn how to use it.

Theorem

Every matrix is row equivalent to exactly one matrix in RREF. (Recall defn. of row equivalent).

Ex: Last time, we saw that $M = \begin{pmatrix} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 6 & -3 & 9 & 3 \\ 0 & 0 & 4 & 1 \end{pmatrix}$

$$\xrightarrow{\text{RREF}} \underbrace{\begin{pmatrix} 1 & -1/2 & 0 & 1/8 \\ 0 & 1 & 0 & 5/4 \\ 0 & 0 & 1 & 1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{Call this } A.}$$ ← Fix this, so:

True/False: ① M r.e. A? True!

② M r.e. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$? False!

Also: ③ When is B (any matrix) r.e. M?

↳ If and only if B's RREF equals A!

• How can we use RREF?

↳ To solve linear systems!

Ex: Solve
$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 \quad - 5x_3 = 10$$

Translation: Find a vector $\langle x_1, x_2, x_3 \rangle$ whose components satisfy all equations!

Coefficient Matrix $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix}$ w/ "Right side vector" $\begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$ $\longrightarrow$ Augmented matrix $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$

To solve: → Note: Two augmented matrices are row equiv. <=> their corresponding systems have the same solutions!

- ① write Augmented matrix
- ② put in REF & decide if there is a solution:

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{} \begin{pmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ 0 & 10 & -10 & 10 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_2]{}$$

$$\begin{pmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ 0 & 0 & \boxed{30} & -30 \end{pmatrix} \longrightarrow \text{what does this tell us about solutions?}$$

REF!

Note: Not unique!

Theorem An augmented REF matrix has no row of the form $[0 \ 0 \ 0 \cdots 0 \ b]$, $b \neq 0$ precisely when the corresponding system has some solution!

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(Cont'd)

So, by theorem, our system has <u>some</u> solution!

③ Put into RREF:

$$\begin{pmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{+2} & -8 & 8 \\ 0 & 0 & \boxed{30} & -30 \end{pmatrix} \xrightarrow[R_3 = \frac{1}{30}R_3]{R_2 = \frac{1}{2}R_2} \begin{pmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{1} & -4 & 4 \\ 0 & 0 & \boxed{1} & -1 \end{pmatrix}$$

$$\xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 + 4R_3} \begin{pmatrix} \boxed{1} & -2 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{pmatrix} \xrightarrow{R_1 = R_1 + 2R_2}$$

$$\begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & \text{RHS} \end{array} \end{array}$$

④ Write as system of eq's:

$$\left\{ \begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + 1x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + 1x_3 &= -1 \end{aligned} \right. \Rightarrow \left. \begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned} \right\}$$

"vector equation"

⑤ Rewrite as vector:

Solution is $x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$,

which describes the <u>point</u> $(1, 0, -1)$ in $\mathbb{R}^3$!

Concl: This example had a solution & it was one point.

Answers "is it Consistent"?

Answers "Is it unique?"

- Sometimes, these answers vary.

Ex: If Augmented matrix has REF

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

then by theorem, row 3 $\Rightarrow$ no solution!

↳ why? Row 3 $\longleftrightarrow$ $0x_1 + 0x_2 + 0x_3 = 2$

$\Longleftrightarrow$ $0 = 2$. (not possible!)

Ex: If Augmented matrix has REF

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem guarantees some solution.

RREF $(R_1 = R_1 - 2R_2)$

$$\begin{pmatrix} \boxed{1} & 0 & 1 & 2 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longleftrightarrow \begin{aligned} x_1 + x_3 &= 2 \\ x_2 + x_3 &= 1 \end{aligned} \Longleftrightarrow \begin{aligned} x_1 &= 2 - x_3 \\ x_2 &= 1 - x_3 \\ x_3 &= \text{anything!} \end{aligned}$$

aka free variable!

So: vector eq

$$x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} + x_3\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} \Rightarrow x_1\begin{pmatrix}1\\0\\0\end{pmatrix} + x_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} - x_3\begin{pmatrix}1\\1\\0\end{pmatrix}$$

Those equations define a <u>line</u> in $\mathbb{R}^3$, so there is <u>a</u> solution but not a <u>unique</u> one!

<u>Note:</u> We can go between <u>linear systems</u> $\longleftrightarrow$ <u>vector equations</u> $\longleftrightarrow$ <u>matrix equations</u>:

<u>Ex:</u>

$$\underbrace{\begin{aligned} 3x_1 + x_2 &= 4 \\ x_1 - x_2 + x_3 &= 1 \\ x_2 - 2x_3 &= 7 \end{aligned}}_{\text{linear system}} \longleftrightarrow \underbrace{x_1 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}}_{\text{vector equation}}$$

$$\updownarrow$$

$$\underbrace{\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}}_{\text{Matrix equation.}}$$

use this for HW1 #4 & #5!