1.

$$\begin{tikzcd}
b_1-b_2+b_3-b_4 \arrow[r, dashed, "T"] & T(b_1-b_2+b_3-b_4) \\
\mathbb{R}^4,\ \text{std} \arrow[r, "T"] \arrow[d] & \mathbb{R}^4,\ \text{std} \arrow[d] \\
\mathbb{R}^4,\ \mathcal{B} \arrow[r, "{[T]_{\mathcal{B}}}"'] & \mathbb{R}^4,\ \mathcal{B}
\end{tikzcd}$$

$T(b_1-b_2+b_3-b_4)$ ← what I want.

must be equal

$-2b_1+b_2-12b_3-3b_4$

$$\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \longmapsto \begin{pmatrix}1 & -1 & 0 & 4\\ 0 & 1 & 1 & -1\\ -3 & 1 & 0 & 8\\ 1 & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \qquad \begin{pmatrix}1+1+0-4\\ 0-1+1+1\\ -3-1+0-8\\ 1-1-2-1\end{pmatrix} = \begin{pmatrix}-2\\1\\-12\\-3\end{pmatrix}$$

$$\boxed{\underline{\text{Ans}}:\ T(b_1-b_2+b_3-b_4) = -2b_1+b_2-12b_3-3b_4}$$

2) We want $A$ = canonical matrix for $T$. We know that

$$A = [T(\vec{e}_1) \mid T(\vec{e}_2)] \quad \text{where} \quad \vec{e}_1 = \langle 1, 0\rangle_{std}, \quad \vec{e}_2 = \langle 0, 1\rangle_{std}.$$

So this problem is similar to 1, but we're tracking $\vec{e}_1, \vec{e}_2$ from top left.

<u>Note</u>: We'll need $[\vec{e}_1]_{\mathcal{B}}$, $[\vec{e}_2]_{\mathcal{B}}$, & using old stuff, $\vec{x} = A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

$\Rightarrow [\vec{x}]_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\vec{x}$. So $A_{\mathcal{B}} = [\vec{b}_1 \mid \vec{b}_2] = \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \Rightarrow A_{\mathcal{B}}^{-1} = \begin{pmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{pmatrix}$.

$\vec{e}_1 \longrightarrow T(\vec{e}_1)$ [what we want]

$[\vec{e}_1]_{\mathcal{B}} = A_{\mathcal{B}}^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1/2\\-1/2\end{pmatrix} \longmapsto \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1/2\\-1/2\end{pmatrix} = \begin{pmatrix}-1/2\\-1/2\end{pmatrix} \longmapsto -\tfrac{1}{2}\vec{b}_1 - \tfrac{1}{2}\vec{b}_2 = -\tfrac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} - \tfrac{1}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$

$\vec{e}_2 \longmapsto [\vec{e}_2]_{\mathcal{B}} = \begin{pmatrix}1/2\\1/2\end{pmatrix}$

$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1/2\\1/2\end{pmatrix} = \begin{pmatrix}3/2\\7/2\end{pmatrix} \longmapsto \tfrac{3}{2}\begin{pmatrix}1\\1\end{pmatrix} + \tfrac{7}{2}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-2\\5\end{pmatrix}$

$T(\vec{e}_2)$

$$\boxed{\underline{\text{Ans}}:\ A = \begin{pmatrix}0 & -2\\ -1 & 5\end{pmatrix}}$$

③ $[T]_{\mathcal{B}\to\mathcal{C}} = \Big[\, [T(\vec{b}_1)]_{\mathcal{C}} \;\Big|\; [T(\vec{b}_2)]_{\mathcal{C}} \;\Big|\; [T(\vec{b}_3)]_{\mathcal{C}} \;\Big|\; [T(\vec{b}_4)]_{\mathcal{C}} \,\Big]$

$= \Big[\, [3\vec{c}_1 - \vec{c}_3]_{\mathcal{C}} \;\Big|\; [\vec{c}_1 - 2\vec{c}_2 + \vec{c}_3]_{\mathcal{C}} \;\Big|\; [\vec{c}_1 - \vec{c}_2 - \vec{c}_3]_{\mathcal{C}} \;\Big|\; [\vec{c}_2 - 3\vec{c}_3]_{\mathcal{C}} \,\Big]$

$$\boxed{\text{Ans:} \quad = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \\ -1 & 1 & -1 & -3 \end{pmatrix}.}$$

④ This is the same as ①:

$$\begin{tikzcd} \vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4 \arrow[d] & -5\vec{c}_1 - 12\vec{c}_2 - 3\vec{c}_3 \\ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \arrow[r, mapsto] & \begin{pmatrix} -5 \\ -12 \\ -3 \end{pmatrix} \arrow[u] \end{tikzcd}$$

$$\begin{pmatrix} 1 & -2 & -4 & 4 \\ -3 & 1 & 0 & 8 \\ 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+2-4-4 \\ -3-1+0-8 \\ 0-1-2+0 \end{pmatrix}$$

$$\boxed{\text{Ans: } T(\vec{b}_1 - \vec{b}_2 + \vec{b}_3 - \vec{b}_4) = -5\vec{c}_1 - 12\vec{c}_2 - 3\vec{c}_3}$$

5.

(a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \xrightarrow[REF]{} \begin{pmatrix} 1 & 1-\lambda \\ 0 & \boxed{1-(2-\lambda)(1-\lambda)} \end{pmatrix}$

Note! This is $-1 \cdot$ char. poly!

(i) $\det(A-\lambda I) = (2-\lambda)(1-\lambda) - 1$

(ii) $(2-\lambda)(1-\lambda)-1=0 \Rightarrow \lambda^2 - 3\lambda + 2 - 1 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 1 = 0$

$\Rightarrow \lambda = \dfrac{3 \pm \sqrt{(-3)^2 - 4(1)(1)}}{2}$

$\Rightarrow \lambda = \dfrac{3 \pm \sqrt{5}}{2}.$

So: • $\lambda_1 = \dfrac{3+\sqrt{5}}{2}$ w/ mult. 1

• $\lambda_2 = \dfrac{3-\sqrt{5}}{2}$ w/ mult. 1.

(iii) • $\lambda = \dfrac{3+\sqrt{5}}{2} \Rightarrow \left( REF(A) \,\vdots\, \vec{0} \right) = \left(\begin{array}{cc:c} 1 & 1-\frac{1}{2}(3+\sqrt{5}) & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{array}{l} x_1 = \left(\frac{1}{2}(3+\sqrt{5}) - 1\right)x_2 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = x_2 \begin{pmatrix} \frac{1}{2}(3+\sqrt{5}) - 1 \\ 1 \end{pmatrix}$

are <u>all</u> my eigenvectors.

• $\lambda = \dfrac{3-\sqrt{5}}{2} \Rightarrow \vec{x} = x_2 \begin{pmatrix} \frac{1}{2}(3-\sqrt{5}) - 1 \\ 1 \end{pmatrix}$ are all eigenvectors.

(iv) $\lambda = \dfrac{3+\sqrt{5}}{2}$: $\{\langle \frac{1}{2}(3+\sqrt{5}) - 1, 1 \rangle\}$ $\quad \lambda = \dfrac{3-\sqrt{5}}{2}$ : $\{\langle \frac{1}{2}(3-\sqrt{5}) - 1, 1 \rangle\}$

3]

5 (cont'd)

(b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix}$

(i) $\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 1$

(ii) $(1-\lambda)^2 - 1 = 0 \Rightarrow (1-\lambda)^2 = 1 \Rightarrow 1-\lambda = 1$ or $1-\lambda = -1$

$\Rightarrow \lambda = 0$ or $\lambda = 2$

[mult = 1] [mult = 1]

(iii) $\underline{\lambda = 0}$: $(A - \lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{array}{l} x_1 = -x_2 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\underline{\lambda = 2}$: $(A - \lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{array}{l} x_1 = x_2 \\ x_2 = x_2 \end{array} \Rightarrow \vec{x} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

(iv) $\underline{\lambda = 0}$: $\{\langle -1, 1 \rangle\}$ $\lambda = 2$: $\{\langle 1, 1 \rangle\}$

5 (Cont'd)

(c) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}$

(i) $\det(A - \lambda I) = (1-\lambda)^2$

(ii) $\lambda = 1$ w/ mult. $= 2$

(iii) $(A - \lambda I \mid \vec{0}) = \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{array}{l} x_2 = 0 \\ x_1 = x_1 \end{array} \Rightarrow \vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(iv) $\{\langle 1, 0 \rangle\}$

5 (cont'd)

(d) $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} -\lambda & 2 \\ -2 & -\lambda \end{pmatrix}$

(i) $\det(A - \lambda I) = \lambda^2 + 4$

(ii) $\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4$

$\Rightarrow \lambda = 2i$ mult. $= 1$

$\lambda = -2i$ mult. $= 1$

(iii) • $\underline{\lambda = 2i}$: $(A - \lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right) \xrightarrow[\text{div. by } -2]{} \left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right)$

$\xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right) \xrightarrow[R_2 = R_2 - iR_1]{} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & \boxed{0} & 0 \end{array}\right)$

$-1 - i(i) = -1 - i^2 = -1 - (-1)$

$\Rightarrow \vec{x} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}$

• $\underline{\lambda = -2i}$: By thm in class, the vec will be conj. of $\lambda = 2i$ case: $\vec{x} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$.

(iv) $\underline{\lambda = 2i}$: $\{\langle -i, 1 \rangle\}$ $\underline{\lambda = -2i}$: $\{\langle i, 1 \rangle\}$

5 (cont'd)

(e) $A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{pmatrix}$

(i) $\det(A-\lambda I) = (1-\lambda)\det\begin{pmatrix} 5-\lambda & 6 \\ 8 & 9-\lambda \end{pmatrix}$

$\qquad - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9-\lambda \end{pmatrix}$

$\qquad + 3\det\begin{pmatrix} 4 & 5-\lambda \\ 7 & 8 \end{pmatrix}$

$= (1-\lambda)\big((5-\lambda)(9-\lambda)-48\big) - 2\big(4(9-\lambda)-42\big) + 3\big(32-7(5-\lambda)\big)$

$= -\lambda^3 + 15\lambda^2 + 18\lambda$

(ii) char poly $= 0 \Rightarrow -\lambda^3 + 15\lambda^2 + 18\lambda = 0 \Rightarrow \underbrace{-\lambda}_{\lambda=0}\underbrace{(\lambda^2 - 15\lambda - 18)}_{\lambda = \frac{15 \pm \sqrt{225+72}}{2} = \frac{15\pm 3\sqrt{33}}{2}} = 0$

$\Rightarrow \lambda = 0$ mult. 1

$\lambda = \dfrac{15+3\sqrt{33}}{2}$ mult. 1

$\lambda = \dfrac{15-3\sqrt{33}}{2}$ mult. 1.

(iii) $\underline{\lambda = 0:}$ $(A-\lambda I \,\vdots\, \vec{0}) \equiv (A \,\vdots\, \vec{0}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right) \xrightarrow[\text{REF}]{} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \vec{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\lambda = \overbrace{\dfrac{15+3\sqrt{33}}{2}}^{\lambda_1}$ : $\vec{x} = x_3 \underbrace{\left\langle \dfrac{15+\sqrt{33}}{33+7\sqrt{33}}, \dfrac{24+4\sqrt{33}}{33+7\sqrt{33}}, 1 \right\rangle^T}_{\vec{v}_1}$

$\lambda = \overbrace{\dfrac{15-3\sqrt{33}}{2}}^{\lambda_2}$ : $\vec{x} = x_3 \underbrace{\left\langle \dfrac{-15+\sqrt{33}}{-33+7\sqrt{33}}, \dfrac{-24+4\sqrt{33}}{-33+7\sqrt{33}}, 1 \right\rangle^T}_{\vec{v}_2}$

(iv)

$\underline{\lambda=0:}$ $\{\langle 1,-2,1\rangle\}$

$\underline{\lambda=\lambda_1:}$ $\{\vec{v}_1\}$

$\underline{\lambda=\lambda_2:}$ $\{\vec{v}_2\}$

5 (cont'd)

(f) $A-\lambda I$ is triangular w/ diagonal entries $5-\lambda$, $5-\lambda$, $-3-\lambda$, and $-3-\lambda$.

(i) $(5-\lambda)(5-\lambda)(-3-\lambda)(-3-\lambda)$

(ii) $\lambda = 5$ w/ mult. = 2

$\lambda = -3$ w/ mult. = 2

(iii) $\underline{\lambda = 5:}$ $(A-\lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cccc|c} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 4 & -2 & 0 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & -8 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow x_3 = 0$ & $x_4 = 0$, So, $\vec{x} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{u}_1} + x_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{u}_2}$.

$\underline{\lambda = -3:}$ $(A-\lambda I \,\vdots\, \vec{0}) = \left(\begin{array}{cccc|c} 8 & 0 & 1 & -1 & 0 \\ 0 & 8 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cccc|c} 1 & 0 & 1/8 & -1/8 & 0 \\ 0 & 1 & 1/2 & -1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \vec{x} = x_3 \underbrace{\begin{pmatrix} -1/8 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{v}_1} + x_4 \underbrace{\begin{pmatrix} 1/8 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}}_{\vec{v}_2}$.

(iv) $\underline{\lambda = 5:}$ $\{\vec{u}_1, \vec{u}_2\}$

$\underline{\lambda = -3:}$ $\{\vec{v}_1, \vec{v}_2\}$

6. Using 5f, we write

$$\lambda_1 = 5 \geq \lambda_2 = 5 \geq \lambda_3 = -3 \geq \lambda_4 = -3$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$\vec{u}_1 \quad \vec{u}_2 \quad \vec{v}_1 \quad \vec{v}_2$$

$$\underbrace{\vec{u}_1 \quad \vec{u}_2}_{\text{order doesn't matter}} \quad \underbrace{\vec{v}_1 \quad \vec{v}_2}_{\text{order doesn't matter}}$$

(a) $P = [\vec{u}_1 \mid \vec{u}_2 \mid \vec{v}_1 \mid \vec{v}_2] = \begin{pmatrix} 1 & 0 & -1/8 & 1/8 \\ 0 & 1 & -1/2 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) True. There are lots of ways to see this, but: $P$ is triangular $\Rightarrow$ $\det(P)$ = prod. of diagonal entries $\Rightarrow \det(P) = 1$ $\Rightarrow$ $P$ invertible $\Leftrightarrow$ cols of $P$ are L.I. by invert. matrix thm.

(c) $D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

(d) $D^k = \begin{pmatrix} 5^k & 0 & 0 & 0 \\ 0 & 5^k & 0 & 0 \\ 0 & 0 & (-3)^k & 0 \\ 0 & 0 & 0 & (-3)^k \end{pmatrix}$ for all $k$. Plug in $k=2$, $k=3$, & $k = 2, 196, 432 \ldots$

(e) $AP = \begin{pmatrix} 5 & 0 & 3/8 & -3/8 \\ 0 & 5 & 3/2 & -3/4 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$ (f) $PD =$ The same matrix from (e)!

(g) B/c $AP = PD$, then mult. on the right by $P^{-1}$ gives $A\underbrace{PP^{-1}}_{=I} = PDP^{-1}$ $\Rightarrow A = PDP^{-1}$.

(h) $A^{1032} = (PDP^{-1})^{1032} \overset{\text{use the hint.}}{=} PD^{1032}P^{-1} \leftarrow$ This is a diagonalization for $A^{1032}$.