Name: ______________________________

MAS 3105 — Homework 4

Directions: Complete the following problems for a homework grade, being sure to adhere to the Homework Policy on the Homework tab of the course webpage

`http://www.math.fsu.edu/~cstover/teaching/sp18_mas3105/`.

Date Due: Thursday, April 26.

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis for $\mathbb{R}^4$. Find $\mathrm{T}(\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3 - \mathbf{b}_4)$ when $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is the linear transformation with $\mathcal{B}$-matrix

$$[\mathrm{T}]_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 1 & 1 & -1 \\ -3 & 1 & 0 & 8 \\ 1 & 1 & -2 & 1 \end{pmatrix}.$$

2. Let $\mathcal{B} = \left\{\mathbf{b}_1 = \begin{pmatrix}1, & 1\end{pmatrix}^{\mathsf{T}}, \mathbf{b}_2 = \begin{pmatrix}-1, & 1\end{pmatrix}^{\mathsf{T}}\right\}$ be a basis for $\mathbb{R}^2$. If $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$[\mathrm{T}]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

find the canonical matrix A corresponding to T. **Hint**: Draw the corresponding commutative diagram!

3. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis for a vector space V and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis for a vector space W. Let $\mathrm{T} : V \to W$ be a linear transformation such that

$$T(\mathbf{b}_1) = 3\mathbf{c}_1 - \mathbf{c}_3 \qquad T(\mathbf{b}_2) = \mathbf{c}_1 - 2\mathbf{c}_2 + \mathbf{c}_3 \qquad T(\mathbf{b}_3) = \mathbf{c}_1 - \mathbf{c}_2 - \mathbf{c}_3 \qquad T(\mathbf{b}_4) = \mathbf{c}_2 - 3\mathbf{c}_3.$$

Find $[\mathrm{T}]_{\mathcal{B}\to\mathcal{C}}$.

4. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ be a basis for $\mathbb{R}^4$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ be a basis for a vector space W. Find $\mathrm{T}(\mathbf{b}_1 - \mathbf{b}_2 + \mathbf{b}_3 - \mathbf{b}_4)$ when $\mathrm{T} : \mathbb{R}^4 \to W$ is the linear transformation with $\mathcal{B}$-to-$\mathcal{C}$-matrix

$$[\mathrm{T}]_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} 1 & -2 & -4 & 4 \\ -3 & 1 & 0 & 8 \\ 0 & 1 & -2 & 0 \end{pmatrix}.$$

5. For each of the following matrices A,

(i) find the characteristic polynomial of A;

(ii) find the eigenvalues of A **with multiplicites**;

(iii) find the eigenvectors corresponding to each eigenvalue from part (ii); and

(iv) compute a basis for the eigenspace of each eigenvalue from part (ii).

(a) $\mathsf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

(b) $\mathsf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(c) $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(d) $\mathsf{A} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

(e) $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(f) $\mathsf{A} = \begin{pmatrix} 5 & 0 & 1 & -1 \\ 0 & 5 & 4 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$ **Hint**: Notice that A is triangular.

6. Consider the matrix

$$\mathsf{A} = \begin{pmatrix} 5 & 0 & 1 & -1 \\ 0 & 5 & 4 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

from question 5(f) above. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the eigenvalues of A (written multiple times for multiplicity ≥ 1 and written in descending order) and let $\mathbf{v}_1, \ldots, \mathbf{v}_4$ be the eigenvectors corresponding to $\lambda_1, \ldots, \lambda_4$.

(a) Write down the 4×4 matrix $\mathsf{P} = \left(\mathbf{v}_1 \,\middle|\, \mathbf{v}_2 \,\middle|\, \mathbf{v}_3 \,\middle|\, \mathbf{v}_4\right)$.

(b) **True or False:** The columns of P are linearly independent. Justify your claim.

(c) Write down the 4×4 diagonal matrix D whose (i,i)-entry equals λ_i and whose (i,j)-entry equals 0 for all $i \neq j$.

Hint: This means that the $(1,1)$-entry is λ_1, the $(2,2)$-entry is λ_2, the $(3,3)$-entry is λ_3, the $(4,4)$-entry is λ_4, and all other entries are 0.

(d) Compute D^2, D^3, and $\mathsf{D}^{2\,196\,432}$. **Hint**: This should require almost no work; do D^2 and find the pattern!

(e) Compute AP.

(f) Compute PD.

(g) Using parts (b), (e), and (f), conclude that $\mathsf{A} = \mathsf{PDP}^{-1}$. **Do not compute P^{-1}!**

This decomposition is called the diagonalization of A, and the process given in parts (a)–(e) is called diagonalizing A. As it happens, not all matrices are diagonalizable, and in fact, an $n \times n$ matrix A is diagonalizable if and only if the sums of the dimensions of its eigenspaces is equal to n. For the matrix A in this problem, you can confirm that this holds by referring back to 5(f) above.

Now, the question remains: *Why* is diagonalization helpful?

(h) Using parts (d) and (g), find a diagonalization for $\mathsf{A}^{1\,032}$ *without doing any work!*.

Hint: $\mathsf{A}^2 = \left(\mathsf{PDP}^{-1}\right)^2 = \left(\mathsf{PDP}^{-1}\right)\left(\mathsf{PDP}^{-1}\right) = \mathsf{PD}\underbrace{\mathsf{P}^{-1}\mathsf{P}}_{(!!!)}\mathsf{DP}^{-1}$.