2.

(a)

(i) $$\begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 2 & 3 & 1 \\ 3 & -7 & -3 & 7 \end{pmatrix} \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 5/2 & 7/2 \\ 0 & 1 & 3/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

either can be a basis for row(A)

says two vecs are basis for col(A) (first two cols of A)

(ii) $\{\langle 1,0,3\rangle^T, \langle -1,2,-7\rangle^T\}$

(iii) 2

(iv) $\{\langle 1,-1,1,3\rangle, \langle 0,2,3,1\rangle\}$ or $\{\langle 1,0,5/2,7/2\rangle, \langle 0,1,3/2,1/2\rangle\}$

(v) 2

(vi) $(A \,\vdots\, \vec{0}) \xrightarrow[\text{RREF}]{} (\text{RREF}(A) \,\vdots\, \vec{0})$, so

$$\left(\begin{array}{cccc|c} 1 & 0 & 5/2 & 7/2 & 0 \\ 0 & 1 & 3/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 &= -5/2\,x_3 - 7/2\,x_4 \\ x_2 &= -3/2\,x_3 - 1/2\,x_4 \\ x_3 &= 1x_3 + 0x_4 \\ x_4 &= 0x_3 + 1x_4 \end{aligned} \Rightarrow \vec{x} = x_3 \begin{pmatrix} -5/2 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -7/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow \{\langle -5/2, -3/2, 1, 0\rangle^T, \langle -7/2, -1/2, 0, 1\rangle^T\}$ is a basis for nul(A)

(vii) 2

(viii) rank(A) = 2; nullity(A) = 2; # cols(A) = 4.

$\Rightarrow$ rank + nullity = # cols !

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2(b)

(i) $$A \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3 rows L.I.

All three cols L.I.

(ii) Basis $= \{\text{col}\,1, \text{col}\,2, \text{col}\,3\}$ ← of A, not RREF(A)!

$= \{\langle -2,4,1,0,3\rangle^T, \langle 1,1,0,2,-2\rangle^T, \langle 3,1,1,2,4\rangle^T\}$

(iii) 3

(iv) Basis $= \{\text{row}\,1, \text{row}\,2, \text{row}\,3\}$ ← of either A or RREF(A)!

$= \{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$ or $\{\langle -2,1,3\rangle, \langle 4,1,1\rangle, \langle 1,0,1\rangle\}$

(v) 3

(vi) $$(A \,\vdots\, \vec{0}) \xrightarrow[\text{RREF}]{} (\text{RREF}(A) \,\vdots\, \vec{0}) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$$

If $A\vec{x} = \vec{0}$ has only the trivial solution, a basis for nul(A) DNE!

(vii) 0 (if nul(A) has no basis, dim(nul(A)) = 0!)

(viii) rank(A) + nullity(A) $\overset{?}{=}$ # cols (A)

$3 + 0 \overset{?}{=} 3$

Yes!

2)

2 (c)

(i) $A \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ rows are L.I.

cols are L.I.

(ii) $\{\langle 1,5,-1,9\rangle^T, \langle 2,6,-2,10\rangle^T, \langle 3,7,3,-11\rangle^T\}$

(iii) 3

(iv) $\{\langle 1,0,0,-2\rangle, \langle 0,1,0,3\rangle, \langle 0,0,1,0\rangle\}$ or

$\{\langle 1,2,3,4\rangle, \langle 5,6,7,8\rangle, \langle -1,-2,3,-4\rangle\}$

(v) 3

(vi) $$(\text{RREF}(A) \,\vdots\, 0) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 2x_4 \\ x_2 = -3x_4 \\ x_3 = 0x_4 \\ x_4 = 1x_4 \end{array} = x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

So, a basis for nul(A) $= \{\langle 2,-3,0,1\rangle^T\}$.

(vii) 1

(viii) rank(A) = 3; nullity(A) = 1; # cols(A) = 4.

$3 + 1 = 4$!

3. (a)

(i) $A^T = \begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & -7 \\ 1 & 3 & -3 \\ 3 & 1 & 7 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(ii) $\{\langle 1, -1, 1, 3\rangle^T, \langle 0, 2, 3, 1\rangle^T\}$

(iii) 2

(iv) $\{\langle 1, 0, 3\rangle, \langle 0, 1, -2\rangle\}$ or $\{\langle 1, 0, 3\rangle, \langle -1, 2, -7\rangle\}$

(v) 2

(vi) $(\text{RREF}(A^T) \mid \vec{0}) = \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = -3x_3 \\ x_2 = 2x_3 \\ x_3 = x_3 \end{array} = x_3 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow$ basis is $\{\langle -3, 2, 1\rangle\}$

(vii) 1

(viii) $\text{rank}(A^T) = 2$; $\text{nullity}(A^T) = 1$; # cols $(A^T) = 3$. ✓

4

3(b)

(i)

$$A^T \xrightarrow[\text{RREF}]{} \begin{pmatrix} 1 & 0 & 0 & 1 & -5/8 \\ 0 & 1 & 0 & 1 & -11/8 \\ 0 & 0 & 1 & -2 & 29/4 \end{pmatrix}$$

(ii)

$\{\langle -2,1,3\rangle^T, \langle 4,1,1\rangle^T, \langle 1,0,1\rangle^T\}$

(iii) 3

(iv) $\{\langle 1,0,0,1,-5/8\rangle, \langle 0,1,0,1,-11/8\rangle, \langle 0,0,1,-2,29/4\rangle\}$ or

$\{\langle -2,4,1,0,3\rangle, \langle 1,1,0,2,-2\rangle, \langle 3,1,1,2,4\rangle\}$

(v) 3

(vi) $$\left(RREF(A^T) \,\vdots\, \vec{0}\right) = \left(\begin{array}{ccccc:c} 1 & 0 & 0 & 1 & -5/8 & 0 \\ 0 & 1 & 0 & 1 & -11/8 & 0 \\ 0 & 0 & 1 & -2 & 29/4 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 &= -x_4 + \tfrac{5}{8}x_5 \\ x_2 &= -x_4 + 11/8\, x_5 \\ x_3 &= 2x_4 - 29/4\, x_5 \\ x_4 &= 1x_4 + 0x_5 \\ x_5 &= 0x_4 + 1x_5 \end{aligned}, \quad \text{so}$$

a basis is $\{\langle -1,-1,2,1,0\rangle^T, \langle 5/8, 11/8, -29/4, 0, 1\rangle^T\}$.

(vii) 2

(viii) rank$(A^T) = 3$; nullity$(A^T) = 2$; #cols$(A^T) = 5$. ✓

3 (c)

(i) $A^T \xrightarrow[RREF]{} \begin{pmatrix} 1 & 0 & 0 & -14/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -11/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(ii) $\{\langle 1,2,3,4\rangle^T, \langle 5,6,7,8\rangle^T, \langle -1,-2,3,-4\rangle^T\}$

(iii) 3

(iv) $\{\langle 1,0,0,-14/3\rangle, \langle 0,1,0,2\rangle, \langle 0,0,1,-11/3\rangle\}$ or

$\{\langle 1,5,-1,9\rangle, \langle 2,6,-2,10\rangle, \langle 3,7,3,-11\rangle\}$

(v) 3

(vi) $(RREF(A^T) \vdots \vec{0}) = \left(\begin{array}{cccc:c} 1 & 0 & 0 & -14/3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -11/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 14/3\, x_4 \\ x_2 = -2x_4 \\ x_3 = 11/3\, x_4 \\ x_4 = 1x_4 \end{array}$, so a

basis is $\{\langle 14/3, -2, 11/3, 1\rangle^T\}$.

(vii) 1

(viii) $\text{rank}(A^T) = 3$; $\text{nullity}(A^T) = 1$; $\#\text{cols}(A^T) = 4$. ✓

4(a)

(i) $D = \mathbb{R}^2$

$CD = \mathbb{R}^5$

(ii) $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ -1 & 1 \\ 2 & 3 \end{pmatrix}$

L.I.

(iii) $\text{range}(T) = \text{col}(A)$

$= \text{span}\{\langle 1, 1, 0, -1, 2\rangle^T, \langle 0, -1, 1, 1, 3\rangle^T\}$

(iv) 2

(v) $\ker(T) = \text{nul}(A)$, so consider $A\vec{x} = \vec{0} \iff (A \,\vdots\, \vec{0})$:

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_1 - x_2 = 0 \\ x_2 = 0 \\ -x_1 + x_2 = 0 \\ 2x_1 + 3x_2 = 0 \end{array} \Rightarrow \vec{x} = \vec{0}.$$

So $\ker(T) = \{\vec{0}\}$. $\Rightarrow$ T is injective!

(vi) $\ker(T) = \{\vec{0}\}$ has no basis, so $\dim = 0$.

4(b)

(i) $D = \mathbb{R}^4$; $CD = \mathbb{R}^2$

(ii)
$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

(iii)
$$\begin{aligned} \text{range}(T) &= \text{Col}(A) \\ &= \text{span}\{\underbrace{\langle 1,0\rangle^T}_{\vec{v}_1}, \underbrace{\langle 0,1\rangle^T}_{\vec{v}_2}, \underbrace{\langle 0,-1\rangle^T}_{=-\vec{v}_2}, \underbrace{\langle -1,0\rangle^T}_{=-\vec{v}_1}\} \\ &= \text{span}\{\langle 1,0\rangle^T, \langle 0,1\rangle^T\}. \end{aligned}$$

(iv) 2

(v) $\ker(T) = \text{nul}(A) \longleftrightarrow (\vec{A} \,\vdots\, \vec{0}) = \left(\begin{array}{cccc:c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right)$

$$\Rightarrow \begin{aligned} x_1 &= 0x_3 + 1x_4 \\ x_2 &= 1x_3 + 0x_4 \\ x_3 &= 1x_3 + 0x_4 \\ x_4 &= 0x_3 + 1x_4 \end{aligned}, \quad \text{so } \ker(T) = \text{span}\{\langle 0,1,1,0\rangle^T, \langle 1,0,0,1\rangle^T\}$$

(vi) 2

4(c)

(i) $D = \mathbb{R}^4$; $CD = \mathbb{R}^4$

(ii) $$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix} \xrightarrow[\text{RREF}]{} I_4$$

all cols L.I.

(iii) range(T) = col(A)

$= \text{span}\{\langle 0,-1,0,1\rangle^T, \langle 0,0,1,-1\rangle^T, \langle 0,0,-1,-1\rangle^T, \langle 1,0,-1,0\rangle^T\}$

(iv) 4

(v) A is 4×4 & r.e. to I_4. By invertible matrix thm, T is injective $\Rightarrow$ $\ker(T) = \text{nul}(A) = \{\vec{0}\}$.

(vi) $\text{nul}(A) = \{\vec{0}\}$ has no basis so dim $= 0$.

5 (a) $S(\vec{x}) = A^T \vec{x}$ where $A^T = \begin{pmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & -1 & 1 & 1 & 3 \end{pmatrix}$.

B/c A^T is 2×5, $S: \mathbb{R}^5 \to \mathbb{R}^2 \Rightarrow$

(i) Dom $= \mathbb{R}^5$ Codom $= \mathbb{R}^2$.

(ii) A^T is canonical matrix

(iii) $\text{range}(S) = \text{Col}(A^T) = \text{span}\{\underbrace{\langle 1,0\rangle^T}_{\vec{v}_1}, \underbrace{\langle 1,-1\rangle^T}_{\vec{v}_1-\vec{v}_2}, \underbrace{\langle 0,1\rangle^T}_{\vec{v}_2}, \underbrace{\langle -1,1\rangle^T}_{-\vec{v}_1+\vec{v}_2}, \underbrace{\langle 2,3\rangle^T}_{2\vec{v}_1+3\vec{v}_2}\}$

$\Rightarrow = \text{span}\{\langle 1,0\rangle^T, \langle 0,1\rangle^T\}$

$= \mathbb{R}^2$.

So range = codom $\Rightarrow$ S is surjective!

(iv) 2

(v) $\ker(S) = \text{nul}(A^T) \longleftrightarrow (A^T \,\vdots\, \vec{0})$, so

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 & 3 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 &= -x_3 + 0x_4 - 5x_5 \\ x_2 &= x_3 + x_4 + 3x_5 \\ x_3 &= x_3 + 0x_4 + 0x_5 \\ x_4 &= 0x_3 + x_4 + 0x_5 \\ x_5 &= 0x_3 + 0x_4 + x_5 \end{aligned}$$

$x_1 = -x_2 + 0x_3 + x_4 - 2x_5$

$= -(x_3 + x_4 + 3x_5) + 0x_3 + x_4 - 2x_5$

Hence,

$\ker(S) = \text{Span}\{\langle -1,1,1,0,0\rangle^T, \langle 0,1,0,1,0\rangle^T, \langle -5,3,0,0,1\rangle^T\}$

(vi) 3

$\boxed{10}$

5(b)

$S(\vec{x}) = A^T\vec{x}$ where $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}$. $= 4\times2$

(i) $D: \mathbb{R}^2$; $CD: \mathbb{R}^4$

(ii) matrix $= A^T$

(iii) range$(S) = \text{col}(A^T)$

$= \text{span}\{\langle 1,0,0,-1\rangle^T, \langle 0,1,-1,0\rangle^T\}$

(iv) 2

(v) $\ker(S) = \text{nul}(A^T) \longleftrightarrow (A^T \vdots \vec{0}) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{array}\right)$

$\Rightarrow$ $x_1 = 0$, $x_2 = 0$. So, $A^T\vec{x} = \vec{0}$ has only the trivial sol'n

$\Rightarrow \ker(S) = \{\vec{0}\}$.

(vi) 0

5(c) $S(\vec{x}) = A^T\vec{x}$ where $A^T = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = 4\times 4.$

(i) $D = \mathbb{R}^4$; $CD = \mathbb{R}^4$

(ii) matrix $= A^T$

(iii) A^T is r.e. to I_4 $\Rightarrow$ all 4 cols are L.I. Hence,

$$\text{range}(S) = \text{Col}(A^T)$$
$$= \text{span}\{\langle 0,0,0,1\rangle^T, \langle -1,0,0,0\rangle^T, \langle 0,1,-1,-1\rangle^T, \langle 1,-1,-1,0\rangle^T\}$$

(iv) 4

(v) As before, A^T = square & r.e. to I_4 $\Rightarrow$ A^T invertible

$\Rightarrow$ S injective $\Rightarrow$ $\ker(S) = \{\vec{0}\}$.

(alternatively: A was invertible by 4(c) $\Rightarrow A^T$ invertible $\Rightarrow \ldots$)

(vi) 0

6. <u>Note:</u> $A \xrightarrow[\text{RREF}]{} \left(\begin{array}{ccccccc} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ L.I.

$$\begin{array}{ccccccc} v_1 & v_2 & \text{LD on } v_1, v_2 & v_3 & \text{LD on } v_1, v_2, v_3 & v_4 & \text{LD on } v_1, v_2, v_3, v_4 \end{array}$$

(a) can use cols 1, 2, 4, & 6 of A, so:

$$C = \begin{pmatrix} 7 & -9 & 5 & -3 \\ -4 & 6 & -2 & -5 \\ 5 & -7 & 5 & 2 \\ -3 & 5 & -1 & -4 \\ 6 & -8 & 4 & 9 \end{pmatrix} \quad 5\times 4$$

(b) using $[\text{RREF}(A) \,\vdots\, \vec{0}\,]$,

$$\begin{aligned} x_1 + \tfrac{13}{2}x_3 + 5x_5 - 3x_7 &= 0 \\ x_2 + \tfrac{11}{2}x_3 + \tfrac{1}{2}x_5 + 2x_7 &= 0 \\ x_4 - \tfrac{11}{2}x_5 + 7x_7 &= 0 \\ x_6 + x_7 &= 0 \end{aligned} \Rightarrow x_3\begin{pmatrix} -13/2 \\ -11/2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5\begin{pmatrix} -5 \\ -1/2 \\ 0 \\ 11/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7\begin{pmatrix} 3 \\ -2 \\ 0 \\ -7 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow N = \begin{pmatrix} -13/2 & -5 & 3 \\ -11/2 & -1/2 & -2 \\ 1 & 0 & 0 \\ 0 & 11/2 & -7 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad 7\times 3$$

(c) $$R = \begin{pmatrix} 1 & 0 & 13/2 & 0 & 5 & 0 & -3 \\ 0 & 1 & 11/2 & 0 & 1/2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -11/2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad 4\times 7$$

(d) Considering $(\text{RREF}(A^T) \,\vdots\, \vec{0}) = \left(\begin{array}{ccccc:c} 1 & 0 & 0 & 0 & -2/11 & 0 \\ 0 & 1 & 0 & 0 & -41/11 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 28/11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, $\longrightarrow$

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6(d) Cont'd

$$\begin{aligned} x_1 &= \tfrac{2}{11}x_5 \\ x_2 &= \tfrac{41}{11}x_5 \\ x_3 &= 0 \\ x_4 &= -\tfrac{28}{11}x_5 \\ x_5 &= x_5 \end{aligned} \quad = x_5 \underbrace{\begin{pmatrix} 2/11 \\ 41/11 \\ 0 \\ -28/11 \\ 1 \end{pmatrix}}_{\vec{v}}.$$

So $M = \vec{v}$. 5×1

(e) $\overset{7\times7}{S} = (\underset{7\times4}{R^T} \,\vdots\, \underset{7\times3}{N}) = \left(\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & -13/2 & -5 & 3 \\ 0 & 1 & 0 & 0 & -11/2 & -1/2 & -2 \\ 13/2 & 11/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 11/2 & -7 \\ 5 & 1/2 & -11/2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ -3 & 2 & 7 & 1 & 0 & 0 & 1 \end{array}\right)$ is 7×7

(f) $\det(S) = \dfrac{-88\,369}{8}$ & S^{-1} exists!

I'm not writing it here. :P

(g) $\overset{5\times5}{T} = (\underset{5\times4}{C} \,\vdots\, \underset{5\times1}{M}) = \left(\begin{array}{cccc|c} 7 & -9 & 5 & -3 & 2/11 \\ -4 & 6 & -2 & -5 & 41/11 \\ 5 & -7 & 5 & 2 & 0 \\ -3 & 5 & -1 & -4 & -28/11 \\ 6 & -8 & 4 & 9 & 1 \end{array}\right)$

(h) $\det(T) = \dfrac{10\,360}{11}$ & T^{-1} exists!