1\. (a) • Is linear (will give details for some later parts)

• domain $= \mathbb{R}^2$

codomain $= \mathbb{R}^2$

range $= \mathbb{R}^2 \leftarrow$ RHS $= \begin{pmatrix} y \\ x \end{pmatrix} = y\begin{pmatrix} 1 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= y\vec{v}_1 + x\vec{v}_2$ where $\vec{v}_1, \vec{v}_2$ L.I.

$\approx \mathbb{R}^2$

• $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

• $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

• T is injective (A has L.I. columns)

• T is surjective (codom(T) = range(T)).

For linear

LHS $= T(c\vec{u} + d\vec{v})$

RHS $= cT(\vec{u}) + dT(\vec{v})$

(b) • is linear

↳ let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then LHS $= T(c\vec{u} + d\vec{v}) =$

$T\begin{pmatrix} cu_1 + dv_1 \\ cu_2 + dv_2 \end{pmatrix} = \begin{pmatrix} cu_1 + dv_1 \\ cu_1 + dv_1 \end{pmatrix}$ & RHS $= cT\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + dT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$

$c\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + d\begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} cu_1 + dv_1 \\ cu_1 + dv_1 \end{pmatrix}$. Hence, LHS = RHS.

• domain $= \mathbb{R}^2$; codomain $= \mathbb{R}^2$; Range $= \mathbb{R}$

↑

$\begin{pmatrix} x \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} = x\vec{v}$, where $\vec{v}$ L.I.

so range $\approx \mathbb{R}$.

• $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

• $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

• Not injective (cols of A are L.D.)

• Not surjective (codom$(T) \neq$ range(T))

(c) • is linear

↳ $LHS = T\begin{pmatrix} cu_1+dv_1 \\ cu_2+dv_2 \end{pmatrix} = \begin{pmatrix} (cu_1+dv_1)+(cu_2+dv_2) \\ cu_1+dv_1 \end{pmatrix}$

$RHS = cT\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + dT\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} cu_1+cu_2 \\ cu_1 \end{pmatrix} + \begin{pmatrix} dv_1+dv_2 \\ dv_1 \end{pmatrix}$

$= \begin{pmatrix} cu_1+dv_1+cu_2+dv_2 \\ cu_1+dv_1 \end{pmatrix} = LHS.$

• Domain $= \mathbb{R}^2$; codom $= \mathbb{R}^2$; range $= \mathbb{R}^2$

• $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

↳ $\begin{pmatrix} x+y \\ x \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \end{pmatrix} = x\vec{v}_1 + y\vec{v}_2$

where $\vec{v}_1, \vec{v}_2$ L.I. So range $= \mathbb{R}^2$.

• $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

• Is injective (A has L.I. cols)

• Is surjective (range (T) = codom (T))

(d) • Is linear

• Domain $= \mathbb{R}^2$; codom $= \mathbb{R}^2$; range $= \mathbb{R}$

• $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

↳ $\begin{pmatrix} 0 \\ y \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = y\vec{v} \Rightarrow$ Range $= \mathbb{R}$

• $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

• Not injective (A has L.D. cols)

• Not surjective (range ≠ codom; also, $\begin{pmatrix} \# \\ y \end{pmatrix}$ not hit if $\# \neq 0$)

(e) • Is linear

• dom $= \mathbb{R}^3$; codom $= \mathbb{R}^2$; range $= \mathbb{R}^2$

• $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$; $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} = x\vec{v}_1 + y\vec{v}_2$ w/ $\vec{v}_1$ & $\vec{v}_2$ L.I.

$\Rightarrow$ range $= \mathbb{R}^2$

• $A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\end{pmatrix}$

• Not injective (A has L.D. Cols)

• is surjective (Codom = range)

Note: This DOESN'T violate the invertible matrix theorem; that only holds for square matrices!

(f) • is linear

• dom $= \mathbb{R}^3$; codom $= \mathbb{R}^3$; range $= \mathbb{R}^2$

• $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\\0\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} = x\vec{v}_1 + y\vec{v}_2$ w/ $\vec{v}_1, \vec{v}_2$ L.I.

$\Rightarrow$ range $\approx \mathbb{R}^2$

• $A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$

• Not injective (A has L.D. Cols)

• Not surjective (Codom $\neq$ range) ← Also, not injective + square $\Rightarrow$ not surjective by inverse matrix thm!

(g) • linear

• dom $= \mathbb{R}^3$; cod $= \mathbb{R}^4$; range $= \mathbb{R}^3$ ← $\begin{pmatrix}x\\y\\z\\0\end{pmatrix} = x\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$= x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3$ w/ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ L.I

$\Rightarrow$ range $\approx \mathbb{R}^3$

• $T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; $T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

• $A = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$

• injective (A has L.I. cols)

• not surjective (range $\neq$ codom)

3)

(h) • is linear

• dom $= \mathbb{R}^4$; codom $= \mathbb{R}^4$; range $= \mathbb{R}^4 \leftarrow \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = w\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$= w\vec{v}_1 + x\vec{v}_2 + y\vec{v}_3 + z\vec{v}_4$ w/ all vectors L.I. $\Rightarrow$ range $\approx \mathbb{R}^4$

• $T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$;

$T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

• $A = I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

• injective (A has L.I. cols)

• surjective (range = codom)

(i) • is linear

• dom $= \mathbb{R}^4$; codom $= \mathbb{R}$; range $= \mathbb{R}$

$\uparrow (w+x+y+z) = w(1) + x(1) + y(1) + z(1)$

$(*) \Rightarrow w\vec{v}_1 + x\vec{v}_2 + y\vec{v}_3 + z\vec{v}_4$ where $\vec{v}_2, \vec{v}_3, \vec{v}_4$ dependent on $\vec{v}_1 \Rightarrow$ range $= c \cdot \vec{v}_1$ for some constant $c \Rightarrow$ range $\approx \mathbb{R}$.

(one L.I. vector in $(*) \Rightarrow$ range $= \mathbb{R}^1 = \mathbb{R}$)

• $T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1$; $T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = 1$;

$T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = 1$; $T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 1$.

• $A = \langle 1, 1, 1, 1 \rangle$

• Not injective (A has L.I. cols)

• surjective (codom = range).

3.
(a) $S: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$T: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow T \circ S$ has matrix $BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $S: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$T: \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \\ y_1 + 2y_2 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}$

$\Rightarrow T \circ S$ has matrix $BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

(c) $S: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_1 + x_2 + x_3 \\ x_1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$T: \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 - y_3 - y_4 \end{pmatrix} \Rightarrow B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$

$\Rightarrow T \circ S$ has matrix $BA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -2 & 0 & -2 \end{pmatrix}$$

4. (a)
- A^3 DNE
- $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$
- $\det(A)$ DNE
- A^{-1} DNE

(b)
- A^3 DNE
- $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$
- $\det(A)$ DNE
- A^{-1} DNE

(c)
- $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $\det(A) = 1$
- $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d)
- $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $\det(A) = 0$
- A^{-1} DNE

(e)
- $A^3 = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$
- $\det(A) = 1(4) - 3(2)$
  $= -2$
- $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$

(f)
- $A^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$
- $A^T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
- $\det(A) = 1$
- $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

(g)
- $A^3 = \begin{pmatrix} 1 & 42 & 239 \\ 0 & 64 & 380 \\ 0 & 0 & 216 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$
- $\det(A) = 24$
- $A^{-1} = \begin{pmatrix} 1 & -1/2 & -1/12 \\ 0 & 1/4 & -5/24 \\ 0 & 0 & 1/6 \end{pmatrix}$

(h)
- $A^3 = \begin{pmatrix} 510 & 624 & 834 \\ 1146 & 1401 & 1872 \\ 2014 & 2462 & 3290 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{pmatrix}$
- $\det(A) = -6$

$A^{-1} = \begin{pmatrix} -7/6 & -1/3 & 1/2 \\ 1/3 & 5/3 & -1 \\ 1/2 & -1 & 1/2 \end{pmatrix}$

(i)
- $A^3 = \begin{pmatrix} -6 & -14 & -3 & 25 \\ 12 & 85 & 70 & -63 \\ -39 & -5 & 23 & 80 \\ 59 & 52 & 5 & -143 \end{pmatrix}$
- $A^T = \begin{pmatrix} 1 & 1 & -2 & 3 \\ -1 & 4 & 0 & 2 \\ 1 & 4 & 1 & 1 \\ 2 & -2 & 3 & -5 \end{pmatrix}$
- $\det(A) = -12$
- $A^{-1} = \begin{pmatrix} 1/3 & 5/6 & -13/6 & -3/2 \\ -1/6 & 19/12 & -41/12 & -\frac{11}{4} \\ 1/6 & \frac{-13}{12} & \frac{35}{12} & \frac{9}{4} \\ 1/6 & \frac{11}{12} & \frac{-25}{12} & \frac{-7}{4} \end{pmatrix}$

Note: $\det(A)=0 \Rightarrow A^{-1}$ DNE.

5. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Write **true** or **false** for each of the following statements about A.

Hint: You may not need to work super-hard for most of these!

(a) $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions. True

(b) The RREF of A is equal to the 3×3 identity matrix I_3. False

(c) The columns of A form a linearly dependent set. True

(d) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one. False

(e) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto. False

(f) The columns of A span $\mathbb{R}^3$. False

(g) The range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ equals its codomain. False

(h) The matrix A^{-1} exists. False

(i) $\det(A) = 7$. False

(j) There exists some $\mathbf{b} \in \mathbb{R}^3$ for which $A\mathbf{x} = \mathbf{b}$ doesn't have a solution. True

(k) A^T is invertible. False

All follows from invertible matrix thm!

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