2.

(a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Note: These solutions are ordered strangely :/

3 (a) Consistent & unique: $x_1 = \frac{4}{3}$ & $x_2 = \frac{5}{3}$

(c) Consistent & unique: $x_1 = 2$, $x_2 = -2$, $x_3 = 1$

(e) Consistent & not unique: $x_1 = $ free, $x_2 = 1 - x_1$, $x_3 = 0$

(by observation, plugging in $x_3 = 0$ yields two identical equations)

(b) Consistent & not unique: $x_1 = $ free, $x_2 = 2x_1 - 1$, $x_3 = 3 - x_1$

(notice: two equations + 3 unknowns)

(d) Inconsistent (by observation, plugging in $x_3 = 0$ yields $\begin{cases} \text{something} = 1 \\ \text{same thing} = -1 \end{cases}$, which is impossible)

(f) Consistent & unique: $x_1 = \frac{-5}{16}$, $x_2 = \frac{17}{16}$, $x_3 = \frac{9}{2}$

4. (a) $x_1\begin{pmatrix}1\\2\end{pmatrix}+x_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

(b) $x_1\begin{pmatrix}1\\2\end{pmatrix}+x_2\begin{pmatrix}0\\1\end{pmatrix}+x_3\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$

(c) $x_1\begin{pmatrix}1\\-1\\1\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}$

(d) $x_1\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

(e) $x_1\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\0\end{pmatrix}+x_3\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$

(f) $x_1\begin{pmatrix}3\\4\\1\end{pmatrix}+x_2\begin{pmatrix}-1\\-4\\-3\end{pmatrix}+x_3\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}7\\-1\\1\end{pmatrix}$

5. All have the form $A\vec{x}=\vec{b}$ where $\vec{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ or $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ & A, $\vec{b}$ are:

(a) $A=\begin{pmatrix}1&1\\2&-1\end{pmatrix}$, $\vec{b}=\begin{pmatrix}3\\1\end{pmatrix}$ (b) $A=\begin{pmatrix}1&0&1\\2&1&0\end{pmatrix}$, $\vec{b}=\begin{pmatrix}3\\1\end{pmatrix}$

(c) $A=\begin{pmatrix}1&1&1\\-1&-1&1\\1&0&0\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ (d) $A=\begin{pmatrix}1&1&1\\-1&-1&1\\0&0&0\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\1\\0\end{pmatrix}$

(e) $A=\begin{pmatrix}1&1&1\\-1&-1&1\\0&0&1\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$ (f) $A=\begin{pmatrix}3&-1&2\\4&-4&1\\1&-3&1\end{pmatrix}$, $\vec{b}=\begin{pmatrix}7\\-1\\1\end{pmatrix}$

6. (a) Consistent & unique: $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \end{pmatrix}$

↑ Here are the RREF matrices for each. (as justification)

(b) Inconsistent: $\begin{pmatrix} 1 & 0 & 5/2 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Implies inconsistent

(c) Consistent & non-unique: $\begin{pmatrix} 1 & 0 & 5/2 & 1/2 \\ 0 & 1 & -3/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

These rows are the "useful equations" which give meaningful info.

This tells us nothing

Three vars

I have 3 vars but 2 meaningful eqs, so non-uniqueness follows.

(d) Consistent & unique: $\begin{pmatrix} 1 & 0 & 0 & -48/19 \\ 0 & 1 & 0 & 44/19 \\ 0 & 0 & 1 & 23/19 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

useful eq's

garbage eq

Compare with (c)

3 vars

3 vars & 3 meaningful eq's ⇒ unique.

(e) Consistent & unique: $\begin{pmatrix} 1 & 0 & 0 & -63/10 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 37/10 \end{pmatrix}$

7. Let $A = [\vec{a}_1 \mid \vec{a}_2 \mid \vec{a}_3]$ & build augmented matrix for eq. $A\vec{x} = \vec{b}$:
$$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 6 \end{pmatrix}.$$
Its RREF is
$$\begin{pmatrix} 1 & 0 & 0 & \boxed{4} \\ 0 & 1 & 0 & \boxed{-5} \\ 0 & 0 & 1 & \boxed{-1} \end{pmatrix},$$

So the system is <u>consistent</u> & the answer is yes!

In particular,
$$\vec{b} = \underline{4}\vec{a}_1 \ \underline{-5}\vec{a}_2 \underline{-} \vec{a}_3.$$

8. The augmented matrix & its RREF are:
$$\begin{pmatrix} -1 & 1 & 3 & -4 \\ 1 & 1 & 2 & -2 \\ 0 & 1 & -4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \boxed{9/13} \\ 0 & 1 & 0 & \boxed{-19/13} \\ 0 & 0 & 1 & \boxed{-8/13} \end{pmatrix}$$

So, $\vec{b} = \underline{9/13}\,\vec{a}_1 \underline{- 19/13}\,\vec{a}_2 \underline{- 8/13}\,\vec{a}_3$ (where $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are the columns of A).

9. <u>True.</u> Using the hint, I want to solve $\vec{w} = s\vec{u} + t\vec{v}$ for s & t. So, $\vec{w} = s\vec{u} + t\vec{v} \iff \begin{pmatrix} a \\ b \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 2 \end{pmatrix}$
$$\iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} s + 0t \\ 0s + 2t \end{pmatrix}$$
$$\iff \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} s \\ 2t \end{pmatrix} \iff \begin{matrix} s = a \\ t = b/2 \end{matrix}.$$

Using later stuff: $\vec{u}$ & $\vec{v}$ are linearly <u>independent</u> so their span is 2-dimensional. The only 2-dim. subspace of $\mathbb{R}^2$ ($\mathbb{R}^2$ b/c vectors have 2 components) is $\mathbb{R}^2$, so <u>every</u> vector in $\mathbb{R}^2$ is in the span of $\vec{u}$ & $\vec{v}$ (including $\vec{w}$!).

10. (a) Here is the RREF version of the augmented matrix for $A\vec{x} = \vec{0}$:

$$\left(\begin{array}{cccc|c} 1 & 3 & -3 & 4 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 9 & -11 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right)$$

Now, the solutions are

$$\begin{aligned} x_1 \quad + 9x_3 - 11x_4 &= 0 \Rightarrow x_1 = -9x_3 + 11x_4 \\ x_2 - 4x_3 + 5x_4 &= 0 \Rightarrow x_2 = 4x_3 - 5x_4 \\ & \qquad\quad\; x_3 = \text{free} \\ & \qquad\quad\; x_4 = \text{free} \end{aligned}$$

Rewriting yields

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -9x_3 + 11x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{pmatrix} = \overset{\text{"s"}}{x_3} \overbrace{\begin{pmatrix} -9 \\ 4 \\ 1 \\ 0 \end{pmatrix}}^{\vec{u}} + \overset{\text{"t"}}{x_4} \overbrace{\begin{pmatrix} 11 \\ -5 \\ 0 \\ 1 \end{pmatrix}}^{\vec{v}}.$$

Now, we rename the vectors $\vec{u}$ & $\vec{v}$ & use lowercase english letters q, r, s, t, etc. for parameters (x_3 & x_4):

$$\boxed{\text{Parametric Vector Form (PVF)}: \ \vec{x} = s\vec{u} + t\vec{v}}$$

(b) $A\vec{x} = \vec{0}$ has <u>only</u> the trivial solution here, so really is no PVF. To be pedantic, can write $\boxed{\vec{x} = t\vec{0}.}$

(c) Here is RREF for augmented matrix of $A\vec{x} = \vec{0}$:

$$\left(\begin{array}{cccccc|c} 1 & 5 & 0 & 8 & 1 & 0 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{aligned} x_1 + 5x_2 \quad + 8x_4 + x_5 &= 0 \\ x_3 - 7x_4 + 4x_5 &= 0 \\ x_6 &= 0 \end{aligned}$$

x_2, x_4, x_5 free

So, $$\boxed{\vec{x} =} \begin{pmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{pmatrix} = \underset{r}{x_2} \overbrace{\begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}^{\vec{u}} + \underset{s}{x_4} \overbrace{\begin{pmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}}^{\vec{v}} + \underset{t}{x_5} \overbrace{\begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}}^{\vec{w}} = \boxed{r\vec{u} + s\vec{v} + t\vec{w}}$$

11. (a) $\left(\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 3 & 6 & -1 \\ 4 & 2 & -3 & 4 \end{array}\right)$

Note: (d) & (e) in wrong order!

(b) The RREF of above matrix is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -56/3 \\ 0 & 1 & 0 & 67/3 \\ 0 & 0 & 1 & -34/3 \end{array}\right) \Rightarrow \boxed{\begin{aligned} x_1 &= -56/3 \\ x_2 &= 67/3 \\ x_3 &= -34/3 \end{aligned}}$$

(c) No. $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is the collection containing only $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$: Because $\vec{b} \neq \vec{a}_1$, $\vec{b} \neq \vec{a}_2$, and $\vec{b} \neq \vec{a}_3$, $\vec{b}$ not in that set!

(e) yes. $\vec{b}$ in span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ iff $A\vec{x} = \vec{b}$ has some solution. By (b), it does!

(d) Three vectors.

(f) ∞-many vectors. $W =$ span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ means every vector of form $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$ is in W

can be any real #.

(g) $\vec{a}_2 = 0\vec{a}_1 + 1\vec{a}_2 + 0\vec{a}_3$. in span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

(h) $\vec{a}_1 - \vec{a}_3 = 1\vec{a}_1 + 0\vec{a}_2 + (-1)\vec{a}_3$ in span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

(i) yes. $A\vec{x} = \vec{0}$ has only the trivial solution. (show this!!)

$\longrightarrow$

11 (Cont'd)

(j) $\vec{u}$ in span $\{\vec{a}_1, \vec{a}_2\}$ $\Leftrightarrow$ there are x_1, x_2 such that

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{u}$$

$$\Leftrightarrow x_1\begin{pmatrix}1\\0\\4\end{pmatrix} + x_2\begin{pmatrix}0\\3\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\h\end{pmatrix}.$$

This corresponds to the system

$$\begin{aligned} x_1 + 0x_2 &= 1 \Rightarrow x_1 = 1 \\ 0x_1 + 3x_2 &= -1 \Rightarrow x_2 = -\tfrac{1}{3}. \\ ③\ 4x_1 + 2x_2 &= h \end{aligned}$$

Using ③: $h = 4x_1 + 2x_2 = 4(1) + 2(-1/3) = \boxed{\frac{16}{3}.}$

(k) $\{\vec{a}_2, \vec{b}, \vec{u}\}$ L.D. iff $A\vec{x} = \vec{0}$ has nontrivial solution, where $A = [\vec{a}_2 | \vec{b} | \vec{u}]$ has columns $\vec{a}_2, \vec{b}, \vec{u}$. So let's put the augmented matrix for $A\vec{x} = \vec{0}$ into REF:

$$\left(\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 4 & h & 0 \end{array}\right) \xrightarrow[\text{REF}]{} \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3h - \frac{3}{2} & 0 \end{array}\right)$$

see next pg for row ops to yield this

This has nontrivial solution iff it has a free variable iff the last row is all 0's. Hence, we need

$$3h - \frac{3}{2} = 0 \Rightarrow \boxed{h = \frac{1}{2}.}$$

Note

we can plug in ($\Rightarrow \vec{u} = \begin{pmatrix}1\\-1\\1/2\end{pmatrix}$) & set up the linear system

$$x_1\vec{a}_2 + x_2\vec{b} = \vec{u}$$

to write $\vec{u}$ as an element of span $\{\vec{a}_2, \vec{b}\}$. What we find:

$$\vec{u} = -\frac{1}{4}\vec{a}_2 + \frac{1}{4}\vec{b}.$$

$$\left(\begin{array}{ccc|c} 0 & 4 & 1 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 4 & h & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 2 & 4 & h & 0 \end{array}\right)$$

$$\xrightarrow[R_3 = R_3 - \frac{2}{3}R_1]{} \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 14/3 & h+\frac{2}{3} & 0 \end{array}\right) \xrightarrow[R_3 = 3R_3]{} \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 14 & 3h+2 & 0 \end{array}\right)$$

$\frac{2}{3}$

$4 - \frac{2}{3}(-1)$ $h - {}^2/_3(-1)$

$$\xrightarrow[R_3 - \frac{14}{4}R_2]{} \left(\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3h-\frac{3}{2} & 0 \end{array}\right)$$

$3h+2 - \frac{14}{4}(1)$