Final Exam Preview

Here's a bit of logistical info about the final.

- There will be 8–10 questions overall, and some will have multiple parts.
- The exam will cover the following:
  - Exam 3 material

    Remember: You needed the invertible matrix theorem on Exam 3 too!
  - HW4 material (but not problem 6)
  - Dot/inner products, norm, distance (between vectors), angles (between vectors)
  - Orthogonality / orthogonal complements

    (definitions; $(\text{row}(\mathsf{A}))^\perp = \text{nul}(\mathsf{A})$; $(\text{col}(\mathsf{A}))^\perp = \text{nul}(\mathsf{A}^\mathsf{T})$; etc.)
  - Symmetric matrices

    (definition; they have orthogonal eigenspaces; they have real eigenvalues; etc.)
- The exam will be 35%–50% "old stuff" (Exam 3) and 50%–65% "new stuff" (stuff after Exam 3)
- You should expect the following question formats:
  - computation questions (e.g. using matrices to solve systems from start to finish)
  - multiple-choice questions
  - True/False questions (which may or may not require justification).

  The True/False questions will mostly look like those from Exam 3 and/or the textbook. Sample questions are included herein.
- For some of the above topics, your review questions will be from other sources:
  - Exam 3

    Make sure you use questions 1–2, 3(vi) as a lesson to read the questions carefully!
  - HW4

    You should all read and understand the solution for question 2!!! I ve seen lots of wrong answers handed in for that one!
  - Textbook problems

    I'll update the website accordingly!

Now, here are some sample questions for the remaining topics that you should be able to answer before the exam.

1. Which of the following scenarios are possible? There may be more than one!

   (i). A is 14×14; and
   $(\lambda^2 + 1)^7$ is a factor of $\det(\mathsf{A} - \lambda\,\mathsf{I})$.

   (ii). W is a subspace of $\mathbb{R}^{12}$; and
   $\dim(W) = 7$; and
   $\dim(W^\perp) = 7$

   (iii). A is 2×2; and
   A has real entries; and
   A has eigenvalues $2i$, $-2i$, and 4.

   (iv). A is 3×5; and
   $\text{nullity}(\mathsf{A}^\mathsf{T}) = 3$; and
   $\text{nullity}(\mathsf{A}) = 2$

   (v). W is a subspace of $\mathbb{R}^{12}$; and
   $\dim(W) = 7$; and
   $\dim(W^\perp) = 7$

   (vi). A is 5×5; and
   $\text{rank}(\mathsf{A}) = \mathbb{R}^3$; and
   $\text{nul}(\mathsf{A})$ is 2-dimensional

   (vii). A is 2×2; and
   A has real entries; and
   A has eigenvalues $2i$ and 4.

   (viii). All of the above

   (ix). None of the above

2. Let $\mathbf{x} = \langle 1, 2, 3, 4, 5, 6 \rangle$ and $\mathbf{y} = \langle 4, -3, 2, -1, -6, 5 \rangle$, and let $W = \text{span}\{\mathbf{x}, \mathbf{y}\}$.

   (a) Find $\mathbf{x} \cdot \mathbf{y}$.

   (b) Find $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$.

   (c) Find the distance $\text{dist}(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$.

   (d) Find the angle $\measuredangle(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$.

   (e) What is $\dim(W)$?

   (f) What is the dimension of the orthogonal complement of W? How do you know?

   (g) Let $\mathbf{u} = \langle u_1, u_2, u_3, u_4, u_5, u_6 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle$. Write down the system of equations that must be satisfied by the components of $\mathbf{u}$ and $\mathbf{v}$ for both $\mathbf{u} \in W^\perp$ and $\mathbf{v} \in W^\perp$ to hold. **DO NOT SOLVE!**

3. (a) Give an example of a non-diagonal 5×5 matrix A with real entries which is symmetric. Justify your claim.

   (b) How many of the eigenvalues to your matrix A are real? Check your answer with `Wolfram|Alpha` (`W|A`).
   Hint: Matrices in `W|A` are lists of rows: For example, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is $\{\{1,2\},\{3,4\}\}$ in `W|A`.

   (c) Compute the angles between the eigenspaces for your matrix A.
   Hint: If you forget "the trick," you can always try *this* thing....

   https://en.wikipedia.org/wiki/Angles_between_flats#Angles_between_subspaces

4. Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required!**

   (a) If H_1 and H_2 are two subspaces of a vector space V, then the collection of all vectors in both H_1 and H_2 is a subspace of V.

   (b) If a square matrix is *diagonal*, then it is symmetric. **Hint**: Recall that A is *diagonal* if every entry not on its main diagonal equals zero.

   (c) If E_1 and E_2 are two eigenspaces for a square matrix A with real entries, then $E_1 \perp E_2$.

   (d) The null space of a 3×6 matrix may be 0-dimensional.

   (e) The rank of a 3×6 matrix may be $\mathbb{R}^3$.

   (f) The null space of a 3×6 matrix may be 4-dimensional.

   (g) The column space of a 3×6 matrix may be 0-dimensional.

   (h) For every matrix A, the linearly independent rows of the matrix $\mathrm{RREF}(\mathsf{A})$ are a basis for $\mathrm{row}(\mathsf{A})$.

   (i) For every real number $r \in \mathbb{R}$, there exists a vector $\mathbf{v}$ such that $\langle 1,2,3,4\rangle \cdot \mathbf{v} = r$.

   (j) The row space of a 6×7 matrix is a subspace of $\mathbb{R}^7$.

   (k) The column space of a 6×7 matrix is a subspace of $\mathbb{R}^7$.

(l) The null space of a 6×7 matrix is a subspace of $\mathbb{R}^7$.

(m) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is a linear transformation $V \to V$.

(n) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then $\det\left(\mathsf{A}_{\mathcal{B}\to\mathcal{C}}\right)$ may equal 0.

(o) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is injective.

(p) If $\mathbf{v}$ is orthogonal to every vector in a basis for a vector space W, then $\mathbf{v} \perp W$.

(q) If $\mathbf{v} \perp W$, then $\mathbf{v}$ is orthogonal to every vector in a basis for a vector space W.

(r) If $\lambda_1 \neq \lambda_2$ are two distinct eigenvalues of a square matrix A, there is at least one vector which is in both the eigenspace for λ_1 and in the eigenspace for λ_2.

(s) There exists a matrix A with real coefficients such that $\mathsf{A} = \mathsf{A}^\mathsf{T}$ and whose characteristic polynomial has a factor which is an irreducible quadratic.

(t) The range of a linear transformation is a subspace of its codomain.

(u) The matrix A is invertible if and only if the kernel of the transformation $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is a 0-dimensional subspace of domain(T).

(v) If H is a subspace of $\mathbb{R}^4$, then there is a 4×4 matrix A such that $H = \mathrm{col}(\mathsf{A})$.

(w) If A is $m \times n$ and $\dim(\mathrm{row}(\mathsf{A})) = m$, then the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is one-to-one.

(x) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto, then $\mathrm{rank}(\mathsf{A}) = m$.

(y) There exist two vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and $\measuredangle(\mathbf{u}, \mathbf{v}) < \pi/2$ (radians).